exercise 22. (fundamental domain for the action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$ - part 2)
Consider the action $\phi$ of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$ from exercise 21:

$$A \mapsto \phi_A, \quad \text{such that} \quad \phi_A(z) := \frac{a \cdot z + c}{b \cdot z + d}, \quad \text{for} \quad A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in PSL_2(\mathbb{Z}).$$

Determine a fundamental domain $F$ for this action.

Hint: Use the results from exercise 21.

exercise 23. (presentation of $PSL_2(\mathbb{Z})$)

a) Find a presentation of $\mathbb{Z}_3 \ast \mathbb{Z}_2$ and draw a corresponding Cayley graph.

b) Find an element $P$ of order 3 in $PSL_2(\mathbb{Z})$, such that $P \cdot -\frac{1}{2} + i \frac{\sqrt{3}}{2} = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$, and an element $S$ of order 2, such that $S \cdot i = i$. Using the results from the previous exercise show that $PSL_2(\mathbb{Z}) = \langle P, S \rangle$.

c) Use the ping-pong lemma to show that $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_3 \ast \mathbb{Z}_2$.

Hint: Let $X = \mathbb{R} \cup \{\infty\} = \partial \mathbb{H}$. Show that $PSL_2(\mathbb{Z})$ operates on $X$ and look at $\langle P \rangle \cdot 0$ and $\langle S \rangle \cdot 0$.

d) Find an embedding of the Cayley graph $PSL_2(\mathbb{Z})$ in $\mathbb{H}$ with the help of the fundamental domain $F$ from the previous exercise.

exercise 24. Let $(X,d)$ be a proper, geodesic, metric space. For a fixed $x_0 \in X$ and an $e \in \text{Ends}(X)$ let

$$\mathcal{V}_n(e) = \{ e' \in \text{Ends}(X) \mid e \text{ and } e' \text{ lie in the same path-connected component of } X \setminus B_n(x_0) \}. $$

For fixed $n$ $\text{Ends}(X)$ is covered by the union of the $(\mathcal{V}_n(e))_e$. Show that:

a) $\text{Ends}(X)$ is Hausdorff.

b) $\text{Ends}(X)$ is a totally disconnected space, i.e. every connected component of $\text{Ends}(X)$ consists of a single point.