This exam consists of three independent problems. You may treat them in the order of your choosing.

If you were not able to solve a question but wish to use the result to solve another one, you are welcome to do so, as long as you indicate it explicitly.

**Notation:** if \((E, d)\) is a metric space, \(x \in E\) and \(r > 0\), we denote by \(B_E(x, r)\) the **open** ball centered at \(x\) with radius \(r\), that is,

\[ B_E(x, r) = \{ y \in E, d(x, y) < r \}. \]

**Reminder:** a useful consequence of the Baire Category Theorem is the following.

**Proposition.** If \(E\) is a Baire space and \(\{F_n\}_{n \geq 1}\) is a sequence of closed subsets such that \(\bigcup_{n \geq 1} F_n = E\), then \(\bigcup_{n \geq 1} F_n^\circ\) is a dense open subset of \(E\).

**Problem 1**

1. Is \(c_0(\mathbb{N}) = \{ \{ u_n \} \in \mathbb{R}^\mathbb{N}, \lim_{n \to \infty} u_n = 0 \}\) complete for the norm \(\| \{ u_n \} \|_\infty = \sup_{n \in \mathbb{N}} |u_n|\)?

2. Is \(C([0, 1], \mathbb{R})\) complete for the norm \(\| f \|_1 = \int_0^1 |f(x)| \, dx\)?
**Problem 2**

Let $E$ and $F$ be Banach spaces. We denote by $B$ the closed ball of radius 1 in $E$, that is, $B = B_E(0, 1)$. A bounded operator $T \in \mathcal{L}(E, F)$ is said *compact* if $T(B)$ is compact.

1. Characterize the Banach spaces $E$ such that the identity map $\text{Id}_E$ is compact.
2. Assume that $T \in \mathcal{L}(E, F)$ has finite-dimensional range. Prove that $T$ is compact.
3. Let $T \in \mathcal{L}(E, F)$ be compact and assume that the range $r(T)$ of $T$ is closed in $F$.
   a. Show the existence of $\rho > 0$ such that $B_{r(T)}(0, \rho) \subset T(B)$.
   b. Prove that $r(T)$ is finite-dimensional.
4. *Integral operators with continuous kernels are compact.*

Let $E = (C([0, 1]), \| \cdot \|_\infty)$. For $\kappa \in C([0, 1]^2)$, we define a linear map $T : E \rightarrow E$ by

$$T(f)(x) = \int_0^1 \kappa(x, y)f(y)\,dy.$$  

a. Prove that $T$ is continuous.

b. Prove that $T$ is compact.

**Problem 3**

1. Let $(E, d)$ and $(F, \delta)$ be metric spaces. Assume $E$ complete and consider a sequence $\{f_n\}_{n \geq 1}$ of continuous maps from $E$ to $F$ that converges pointwise to $f : E \rightarrow F$.
   a. Consider, for $n \geq 1$ and $\varepsilon > 0$, the set
      $$F_{n,\varepsilon} = \{x \in E \text{ s. t. } \forall p \geq n, \delta(f_n(x), f_p(x)) \leq \varepsilon\}.$$  
      Show that $\Omega_\varepsilon = \bigcup_{n \geq 1} F_{n,\varepsilon}$ is a dense open subset of $E$.
   b. Show that every point $x_0 \in \Omega_\varepsilon$ has a neighborhood $\mathcal{N}$ such that
      $$\forall x \in \mathcal{N}, \delta(f(x_0), f(x)) \leq 3\varepsilon.$$  
   c. Prove that $f$ is continuous at every point of $\Omega = \bigcap_{n \geq 1} \Omega_{\frac{1}{n}}$ and that $\overline{\Omega} = E$.

2. *Application:* let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Show that its derivative $f'$ is continuous on a dense subset of $\mathbb{R}$. 