MATH 113 - ANALYSIS
SPRING 2015
IN-CLASS MIDTERM

ELEMENTS OF SOLUTION

Notation: if (E,d) is a metric space, $x \in E$ and $r > 0$, we denote by $B_E(x,r)$ the *open* ball centered at x with radius r, that is,

$$B_E(x,r) = \{y \in E \,,\ d(x,y) < r\}.$$

Reminder: a useful consequence of the Baire Category Theorem is the following.

Proposition. *If E is a Baire space and $\{F_n\}_{n\geq 1}$ is a sequence of closed subsets such that $\bigcup_{n\geq 1} F_n = E$, then $\bigcup_{n\geq 1} \overset{\circ}{F_n}$ is a dense open subset of E.*

Problem 1

1. Is $c_0(\mathbb{N}) = \big\{\{u_n\} \in \mathbb{R}^{\mathbb{N}} \,,\ \lim_{n\to\infty} u_n = 0\big\}$ complete for the norm $\|\cdot\|_\infty$?

Yes. Note that it is enough to prove that $c_0(\mathbb{N})$ is closed in $\ell^\infty(\mathbb{N})$, which is complete for the given norm. One may also proceed directly: let $\{u^p\}_{p\in\mathbb{N}}$ be a Cauchy sequence in $c_0(\mathbb{N})$. For $\varepsilon > 0$, there exists a rank N_ε such that $\|u^p - u^q\|_\infty < \frac{\varepsilon}{2}$ for $p, q \geq N_\varepsilon$, that is,

$$(\dagger) \qquad \forall n \in \mathbb{N} \,,\ |u_n^p - u_n^q| < \frac{\varepsilon}{2}.$$

This means that given n fixed, the sequence $\{u_n^p\}_{p\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ complete. Denote $u_n = \lim_{p\to\infty} u_n^p$. We shall prove that

(1) the sequence u belongs to $c_0(\mathbb{N})$,
(2) the convergence occurs for the norm $\|\cdot\|$.

(1) To see that u vanishes at infinity, observe that the Triangle Inequality gives

$$|u_n| \leq |u_n^p| + |u_n - u_n^p|.$$

Fix $p > N_\varepsilon$ and let $q \to \infty$ in $(\dagger)$ to get $|u_n - u_n^p| \leq \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$. Since $u^p \in c_0(\mathbb{N})$, there exists N'_ε such that $n > N'_\varepsilon$ implies $|u_n^p| < \frac{\varepsilon}{2}$ which guarantees $|u_n| < \varepsilon$.
(2) As before, fix $p > N_\varepsilon$, let $q \to \infty$ in $(\dagger)$ and note that N_ε does not depend on n to see that the convergence is uniform.

2. Is $C([0,1],\mathbb{R})$ complete for the norm $\|f\|_1 = \displaystyle\int_0^1 |f(x)|\,dx$?

No. Consider for instance (= draw a picture of) the sequence of continuous functions f_n where

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x \le \frac{1}{2} \\ 1 & \text{if} \quad x \ge \frac{1}{2} + \frac{1}{n} \end{cases}$$

and f_n is affine on $(\frac{1}{2}, \frac{1}{2} + \frac{1}{n})$. Check that $\{f_n\}_{n\in\mathbb{N}}$

- is Cauchy with respect to $\|\cdot\|_1$;
- converges pointwise to the discontinuous function f that is constantly 0 on $[0, \frac{1}{2}]$ and constantly 1 on $(\frac{1}{2}, 1]$.

Prove that $\lim\limits_{n\to\infty} \|f_n - f\|_1 = 0$ to conclude.

Problem 2

Let E and F be Banach spaces. We denote by $\mathbb{B}$ the closed ball of radius 1 in E, that is, $\mathbb{B} = \overline{B_E(0,1)}$. A bounded operator $T \in \mathcal{L}(E,F)$ is said *compact* if $\overline{T(\mathbb{B})}$ is compact. The range of T is denoted by $r(T)$.

1. Characterize the Banach spaces E such that the identity map Id_E is compact.

Riesz's Theorem asserts that Id_E is compact if and only if E is finite-dimensional.

2. Let $T \in \mathcal{L}(E,F)$ with $r(T)$ finite-dimensional. Prove that T is compact.

By the assumption on $r(T)$, it suffices to prove that $\overline{T(\mathbb{B})}$ is closed and bounded. Closedness holds by definition. Boundedness follows from the continuity of T: by definition of the operator norm, $T(\mathbb{B}) \subset B_{r(T)}(0, \|T\|)$ so $\overline{T(\mathbb{B})} \subset \overline{B_{r(T)}(0, \|T\|)}$.

3. Let $T \in \mathcal{L}(E,F)$ be compact and assume that $r(T)$ of T is closed in F.

a. Show the existence of $\rho > 0$ such that $B_{r(T)}(0,\rho) \subset T(\mathbb{B})$.

The operator T induces a surjective continuous linear map $\tilde{T} : E \longrightarrow r(T)$. Since $r(T)$ is closed in F Banach, it is complete so the Open Mapping Theorem applies. Consider for instance the open ball $B_E(0,1)$. Since, $\tilde{T}$ is open, $\tilde{T}(B_E(0,1))$ is an open subset of $r(T)$ that contains 0_F so it must contain a ball centered at 0_F, say

$$B_{r(T)}(0,\rho) \subset \tilde{T}(B_E(0,1)) \subset T(\mathbb{B}).$$

b. Prove that $r(T)$ is finite-dimensional.

Taking closures in the previous inclusion, the closed ball $\overline{B_{r(T)}(0,\rho)}$ is closed in $\overline{T(\mathbb{B})}$, compact by assumption, hence compact itself. Since the dilation by ρ^{-1} is continuous, it follows that $\overline{B_{r(T)}(0,1)}$ is compact, so that Riesz's Theorem implies that $r(T)$ is finite-dimensional.

4. Let $E = (C([0,1]), \|\cdot\|_\infty)$. For $\kappa \in C([0,1]^2)$, we define a linear map $T : E \longrightarrow E$ by

$$T(f)(x) = \int_0^1 \kappa(x,y) f(y)\, dy.$$

a. Prove that T is continuous.

The kernel κ is continuous on the compact $[0,1]^2$ so it is bounded and one can verify that $\|\kappa\|_\infty$ is a Lipschitz constant for T.

b. Prove that T is compact.

The same arguments as in **2.** show that $\overline{T(\mathbb{B})}$ is closed and bounded. By Arzelà-Ascoli, it suffices to prove that $T(\mathbb{B})$ is equicontinuous. This follows from the uniform continuity of κ on the compact $[0,1]^2$: for $0 \leq x, z \leq 1$ and $f \in \mathbb{B}$,

$$|T(f)(x) - T(f)(z)| \leq \|f\|_\infty \int_0^1 |\kappa(x,y) - \kappa(z,y)|\, dy.$$

Since κ is uniformly continuous, there exists $\delta > 0$ such that $|x - z| < \delta$ implies that $|\kappa(x,y) - \kappa(z,y)| < \varepsilon$ for all x, y, z such that $|x - z| < \delta$. For such x and z, we get $|T(f)(x) - T(f)(z)| \leq \varepsilon$, so the family $\{T(f)\ ,\ f \in \mathbb{B}\}$ is equicontinuous.

Problem 3

1. Let (E, d) and (F, δ) be metric spaces. Assume E complete and consider a sequence $\{f_n\}_{n\geq 1}$ of continuous maps from E to F that converges pointwise to $f : E \longrightarrow F$.

a. Consider, for $n \geq 1$ and $\varepsilon > 0$, the set $F_{n,\varepsilon} = \{x \in E\ ,\ \forall p \geq n\ ,\ \delta(f_n(x), f_p(x)) \leq \varepsilon\}$. Show that $\Omega_\varepsilon = \bigcup_{n\geq 1} \overset{\circ}{F}_{n,\varepsilon}$ is a dense open subset of E.

According to the consequence of the Baire Category Theorem recalled above, it suffices to prove that the sets $F_{n,\varepsilon}$ are closed and cover E. For given n and p, the set $\{x \in E\ ,\ \delta(f_n(x), f_p(x)) \leq \varepsilon\}$ is closed as the inverse image of $[0, \varepsilon]$, closed, under the map $x \mapsto \delta(f_n(x), f_p(x))$, continuous as composed of continuous functions. Taking the intersection over $p \geq n$ gives $F_{n,\varepsilon}$ closed. That the union of these sets covers E follows from the pointwise convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$.

b. Show that every point $x_0 \in \Omega_\varepsilon$ has a neighborhood $\mathcal{N}$ such that

$$\forall x \in \mathcal{N}\ ,\ \delta(f(x_0), f(x)) \leq 3\varepsilon.$$

Let n be such that $x_0 \in \overset{\circ}{F}_{n,\varepsilon}$. Since $\overset{\circ}{F}_{n,\varepsilon}$ is open and f_n is continuous, there exists a neighborhood $\mathcal{N}$ of x_0 included in $\overset{\circ}{F}_{n,\varepsilon}$ such that

$$\delta(f_n(x_0), f_n(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N}.$$

Since $\mathcal{N} \subset \overset{\circ}{F}_{n,\varepsilon}$, we have

$$\delta(f_n(x), f_p(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N} \text{ and } p \geq n.$$

Letting $p \to \infty$ in this inequality, we get

$$\delta(f_n(x), f(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N}.$$

Now, by the triangle inequality,

$$\begin{aligned} \delta(f(x), f(x_0)) &\leq \delta(f(x), f_n(x)) + \delta(f_n(x), f_n(x_0)) + \delta(f_n(x_0), f(x_0)) \\ &\leq \varepsilon + \varepsilon + \varepsilon \end{aligned}$$

for all $x \in \mathcal{N}$.

c. Prove that f is continuous at every point of $\Omega = \bigcap_{n\geq 1} \Omega_{\frac{1}{n}}$ and that $\overline{\Omega} = E$.

Let $x_0 \in \Omega$ and $\varepsilon > 0$. Fix n such that $\frac{1}{n} < \frac{\varepsilon}{3}$. By the previous result, there is a neighborhood $\mathcal{N}$ of x_0 such that $\delta(f(x), f(x_0)) \leq \varepsilon$ for all $x \in \mathcal{N}$, which proves continuity of f at x_0. The fact that Ω is dense in E follows from **a.** and the Baire Category Theorem.

2. Let f be differentiable on $\mathbb{R}$. Show that f' is continuous on a dense set.

Apply the previous result to the sequence $f_n : x \mapsto \frac{f\left(x+\frac{1}{n}\right)-f(x)}{1/n}$.