**Notation**: if \((E, d)\) is a metric space, \(x \in E\) and \(r > 0\), we denote by \(B_E(x, r)\) the open ball centered at \(x\) with radius \(r\), that is,

\[
B_E(x, r) = \{y \in E, d(x, y) < r\}.
\]

**Reminder**: a useful consequence of the Baire Category Theorem is the following.

**Proposition.** If \(E\) is a Baire space and \(\{F_n\}_{n \geq 1}\) is a sequence of closed subsets such that

\[
\bigcup_{n \geq 1} F_n = E,
\]

then \(\bigcup_{n \geq 1} F_n\) is a dense open subset of \(E\).

**Problem 1**

1. Is \(c_0(\mathbb{N}) = \{\{u_n\} \in \mathbb{R}^\mathbb{N}, \lim_{n \to \infty} u_n = 0\}\) complete for the norm \(\| \cdot \|_\infty\)?

Yes. Note that it is enough to prove that \(c_0(\mathbb{N})\) is closed in \(\ell^\infty(\mathbb{N})\), which is complete for the given norm. One may also proceed directly: let \(\{u^p\}_{p \in \mathbb{N}}\) be a Cauchy sequence in \(c_0(\mathbb{N})\). For \(\varepsilon > 0\), there exists a rank \(N_\varepsilon\) such that \(\|u^p - u^q\|_\infty < \frac{\varepsilon}{2}\) for \(p, q \geq N_\varepsilon\), that is,

\[
(\dagger) \quad \forall n \in \mathbb{N}, |u^p_n - u^q_n| < \frac{\varepsilon}{2}.
\]

This means that given \(n\) fixed, the sequence \(\{u^p_n\}_{p \in \mathbb{N}}\) is Cauchy in \(\mathbb{R}\) complete. Denote \(u_n = \lim_{p \to \infty} u^p_n\). We shall prove that

1. the sequence \(u\) belongs to \(c_0(\mathbb{N})\),
2. the convergence occurs for the norm \(\| \cdot \|\).

(1) To see that \(u\) vanishes at infinity, observe that the Triangle Inequality gives

\[
|u_n| \leq |u^p_n| + |u_n - u^p_n|.
\]

Fix \(p > N_\varepsilon\) and let \(q \to \infty\) in \((\dagger)\) to get \(|u_n - u^p_n| \leq \frac{\varepsilon}{2}\) for all \(n \in \mathbb{N}\). Since \(u^p \in c_0(\mathbb{N})\), there exists \(N^*_{\varepsilon}\) such that \(n > N^*_{\varepsilon}\) implies \(|u^p_n| < \frac{\varepsilon}{2}\) which guarantees \(|u_n| < \varepsilon\).

(2) As before, fix \(p > N_\varepsilon\), let \(q \to \infty\) in \((\dagger)\) and note that \(N_\varepsilon\) does not depend on \(n\) to see that the convergence is uniform.
2. Is \( C([0,1], \mathbb{R}) \) complete for the norm \( \| f \|_1 = \int_0^1 |f(x)| \, dx \)?

No. Consider for instance (= draw a picture of) the sequence of continuous functions \( f_n \) where
\[
f_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{2} + \frac{1}{n} \end{cases}
\]
and \( f_n \) is affine on \( \left( \frac{1}{2}, \frac{1}{2} + \frac{1}{n} \right) \). Check that \( \{ f_n \}_{n \in \mathbb{N}} \)
- is Cauchy with respect to \( \| \cdot \|_1 \);
- converges pointwise to the discontinuous function \( f \) that is constantly 0 on \( [0, \frac{1}{2}] \)
  and constantly 1 on \( \left( \frac{1}{2}, 1 \right) \).

Prove that \( \lim_{n \to \infty} \| f_n - f \|_1 = 0 \) to conclude.

\[\text{Problem 2}\]

Let \( E \) and \( F \) be Banach spaces. We denote by \( \mathbb{B} \) the closed ball of radius 1 in \( E \), that is, \( \mathbb{B} = \overline{B}_E(0,1) \). A bounded operator \( T \in \mathcal{L}(E,F) \) is said \textit{compact} if \( \overline{T(\mathbb{B})} \) is compact. The range of \( T \) is denoted by \( r(T) \).

1. Characterize the Banach spaces \( E \) such that the identity map \( \text{Id}_E \) is compact.

Riesz’s Theorem asserts that \( \text{Id}_E \) is compact if and only if \( E \) is finite-dimensional.

2. Let \( T \in \mathcal{L}(E,F) \) with \( r(T) \) finite-dimensional. Prove that \( T \) is compact.

By the assumption on \( r(T) \), it suffices to prove that \( \overline{T(\mathbb{B})} \) is closed and bounded. Closed-ness holds by definition. Boundedness follows from the continuity of \( T \): by definition of the operator norm, \( T(\mathbb{B}) \subset B_{r(T)}(0,\| T \|) \) so \( \overline{T(\mathbb{B})} \subset B_{r(T)}(0,\| T \|) \).

3. Let \( T \in \mathcal{L}(E,F) \) be compact and assume that \( r(T) \) of \( T \) is closed in \( F \).

a. Show the existence of \( \rho > 0 \) such that \( B_{r(T)}(0,\rho) \subset T(\mathbb{B}) \).

The operator \( T \) induces a surjective continuous linear map \( \tilde{T} : E \to r(T) \). Since \( r(T) \) is closed in \( F \) Banach, it is complete so the Open Mapping Theorem applies. Consider for instance the open ball \( B_E(0,1) \). Since, \( \tilde{T} \) is open, \( \tilde{T}(B_E(0,1)) \) is an open subset of \( r(T) \) that contains \( 0_F \) so it must contain a ball centered at \( 0_F \), say \( B_{r(T)}(0,\rho) \subset \tilde{T}(B_E(0,1)) \subset T(\mathbb{B}) \).

b. Prove that \( r(T) \) is finite-dimensional.

Taking closures in the previous inclusion, the closed ball \( \overline{B_{r(T)}(0,\rho)} \) is closed in \( \overline{T(\mathbb{B})} \), compact by assumption, hence compact itself. Since the dilation by \( \rho^{-1} \) is continuous, it follows that \( \overline{B_{r(T)}(0,1)} \) is compact, so that Riesz’s Theorem implies that \( r(T) \) is finite-dimensional.
4. Let \( E = (C([0, 1]), \| \cdot \|_{\infty}) \). For \( \kappa \in C([0, 1]^2) \), we define a linear map \( T : E \to E \) by
\[
T(f)(x) = \int_0^1 \kappa(x, y)f(y) \, dy.
\]

a. Prove that \( T \) is continuous.
The kernel \( \kappa \) is continuous on the compact \([0, 1]^2\) so it is bounded and one can verify that \( \| \kappa \|_{\infty} \) is a Lipschitz constant for \( T \).

b. Prove that \( T \) is compact.
The same arguments as in 2. show that \( \overline{T(B)} \) is closed and bounded. By Arzelà-Ascoli, it suffices to prove that \( T(B) \) is equicontinuous. This follows from the uniform continuity of \( \kappa \) on the compact \([0, 1]^2\): for \( 0 \leq x, z \leq 1 \) and \( f \in B \),
\[
|\kappa(x, y) - \kappa(z, y)| < \varepsilon
\]
Since \( \kappa \) is uniformly continuous, there exists \( \delta > 0 \) such that \( |x - z| < \delta \) implies that \( |\kappa(x, y) - \kappa(z, y)| < \varepsilon \) for all \( x, y, z \) such that \( |x - z| < \delta \). For such \( x \) and \( z \), we get \( |T(f)(x) - T(f)(z)| \leq \varepsilon \), so the family \( \{T(f) : f \in B\} \) is equicontinuous.

**Problem 3**

1. Let \((E, d)\) and \((F, \delta)\) be metric spaces. Assume \( E \) complete and consider a sequence \( \{f_n\}_{n \geq 1} \) of continuous maps from \( E \) to \( F \) that converges pointwise to \( f : E \to F \).

   a. Consider, for \( n \geq 1 \) and \( \varepsilon > 0 \), the set \( F_{n, \varepsilon}^* = \{x \in E : \forall p \geq n, \delta(f_n(x), f_p(x)) \leq \varepsilon\} \). Show that \( \Omega_\varepsilon = \bigcup_{n \geq 1} F_{n, \varepsilon}^* \) is a dense open subset of \( E \).

   According to the consequence of the Baire Category Theorem recalled above, it suffices to prove that the sets \( F_{n, \varepsilon}^* \) are closed and cover \( E \). For given \( n \) and \( p \), the set \( \{x \in E : \delta(f_n(x), f_p(x)) \leq \varepsilon\} \) is closed as the inverse image of \([0, \varepsilon]\), closed, under the map \( x \mapsto \delta(f_n(x), f_p(x)) \), continuous as composed of continuous functions. Taking the intersection over \( p \geq n \) gives \( F_{n, \varepsilon}^* \) closed. That the union of these sets covers \( E \) follows from the pointwise convergence of the sequence \( \{f_n\}_{n \in \mathbb{N}} \).

   b. Show that every point \( x_0 \in \Omega_\varepsilon \) has a neighborhood \( \mathcal{N} \) such that
\[
\forall x \in \mathcal{N}, \delta(f(x_0), f(x)) \leq 3\varepsilon.
\]

Let \( n \) be such that \( x_0 \in F_{n, \varepsilon}^* \). Since \( F_{n, \varepsilon}^* \) is open and \( f_n \) is continuous, there exists a neighborhood \( \mathcal{N} \) of \( x_0 \) included in \( F_{n, \varepsilon}^* \) such that
\[
\delta(f_n(x_0), f_n(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N}.
\]
Since $\mathcal{N} \subset F_{n,\varepsilon}$, we have
\[ \delta(f_n(x), f_p(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N} \text{ and } p \geq n. \]
Letting $p \to \infty$ in this inequality, we get
\[ \delta(f_n(x), f(x)) \leq \varepsilon \quad \text{for all } x \in \mathcal{N}. \]
Now, by the triangle inequality,
\[ \delta(f(x), f(x_0)) \leq \delta(f(x), f_n(x)) + \delta(f_n(x), f(x_0)) + \delta(f_n(x_0), f(x_0)) \leq \varepsilon + \varepsilon + \varepsilon \]
for all $x \in \mathcal{N}$.

**c. Prove that $f$ is continuous at every point of $\Omega = \bigcap_{n \geq 1} \Omega_{\frac{1}{n}}$ and that $\overline{\Omega} = E$.**

Let $x_0 \in \Omega$ and $\varepsilon > 0$. Fix $n$ such that $\frac{1}{n} < \frac{\varepsilon}{3}$. By the previous result, there is a neighborhood $\mathcal{N}$ of $x_0$ such that $\delta(f(x), f(x_0)) \leq \varepsilon$ for all $x \in \mathcal{N}$, which proves continuity of $f$ at $x_0$. The fact that $\Omega$ is dense in $E$ follows from a. and the Baire Category Theorem.

**2. Let $f$ be differentiable on $\mathbb{R}$. Show that $f'$ is continuous on a dense set.**

Apply the previous result to the sequence $f_n : x \mapsto \frac{f(x + \frac{1}{n}) - f(x)}{1/n}$. 