The goal of this problem is to give a proof of the following density result.

**Theorem** (Weierstrass). *Every continuous function on a segment of the real line is the uniform limit of a sequence of polynomial functions.*

0. The theorem asserts in particular that the family of functions \( \{ x \mapsto x^n \}_{n \in \mathbb{N}} \) is a topological basis of \((C([0,1]), \| \cdot \|_{\infty})\). Is it an algebraic basis?

No: linear combinations of monomials are smooth while some continuous functions fail to be differentiable.

Let \( \mathcal{E} \) be the space of continuous and compactly supported complex-valued functions on \( \mathbb{R} \). For \( f, g \in \mathcal{E} \), let \( f \ast g \) denote the *convolution product of \( f \) and \( g \), defined by*

\[
f \ast g(x) = \int_{\mathbb{R}} f(t)g(x-t) \, dt.
\]

1. Verify that \((\mathcal{E}, +, \ast)\) is an algebra. Is it unital?

The verification is routine, using Fubini and changes of variables. Note that \( \text{supp}(f \ast g) \subset \text{supp}(f) + \text{supp}(g) \). Assume that \((\mathcal{E}, +, \ast)\) is unital. Then, there exists a continuous function \( f \) such that \( f \ast g = g \) for all \( g \in \mathcal{E} \). In particular, the relation \( f \ast g(0) = g(0) \) implies that \( \int_{\mathbb{R}} f(t)h(t) \, dt = h(0) \) for any \( h \in \mathcal{E} \). Since \( f \) cannot be identically zero, assume that it takes a positive value at \( x_0 \neq 0 \). Then there exists \( \delta > 0 \) such that \( 0 < x_0 - \delta \) and \( f \) only takes positive values on \( I = [x_0 - \delta, x_0 + \delta] \). Consider \( h \) supported in \( I \), non-negative and not identically zero. Then \( h(0) = 0 \neq \int_{\mathbb{R}} f(t)h(t) \, dt \), which contradicts the assumption on \( f \). Therefore \( f \) must vanish everywhere except perhaps at 0, but since it must be continuous, it is constantly zero.
Definition. An approximate unit in $\mathcal{E}$ is a sequence $\{\chi_n\}_{n \geq 1}$ such that for any $f$ in $\mathcal{E}$, the sequence $\{\chi_n \ast f\}$ converges uniformly to $f$.

2. Prove that a sequence of non-negative functions $\alpha_n$ in $\mathcal{E}$ such that

$$\forall n \geq 1 \ , \ \int \alpha_n(t) \, dt = 1 \quad \text{and} \quad \forall A > 0 \ , \ \lim_{n \to \infty} \int_{|t| \geq A} \alpha_n(t) \, dt = 0$$

is an approximate unit.

Let $f \in \mathcal{E}$. Since $f$ is continuous and compactly supported, it is uniformly continuous. Fix $\varepsilon > 0$ and let $\eta > 0$ be such that $|x - y| < \eta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Then, if $n$ is large enough so that $\int_{|t| \geq \eta} \alpha_n(t) \, dt < \varepsilon$,

$$|f \ast \alpha_n(x) - f(x)| = \left| \int_{\mathbb{R}} (f(x - t) - f(x)) \alpha_n(t) \, dt \right| \leq \int_{|t| > \eta} |f(x - t) - f(x)| \alpha_n(t) \, dt + \int_{-\eta}^{\eta} |f(x - t) - f(x)| \alpha_n(t) \, dt$$

$$< 2\|f\|_\infty \varepsilon + \varepsilon \int_{\mathbb{R}} \alpha_n(t) \, dt = (2\|f\|_\infty + 1)\varepsilon,$$

which can be made arbitrarily small, independently of $x$.

3. Define, for $n \geq 1$, $a_n = \int_{-1}^{1} (1 - t^2)^n \, dt$ and $p_n : t \mapsto \begin{cases} \frac{(1 - t^2)^n}{a_n} & \text{if } |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Show that $\{p_n\}_{n \geq 1}$ is an approximate unit in $\mathcal{E}$.

The non-negativity and normalization are immediate. Note that $\int_{|t| \geq A} p_n(t) \, dt = 0$ if $A \geq 1$ and that

$$a_n = 2 \int_{0}^{1} (1 - t^2)^n \, dt \geq 2 \int_{0}^{1} (1 - t)^n \, dt = \frac{2}{n + 1}.$$

For $0 < A < 1$ and $n \geq 1$, we see that

$$\int_{|t| \geq A} p_n(t) \, dt = \frac{2}{a_n} \int_{A}^{1} (1 - t^2)^n \, dt$$

$$\leq \frac{2}{a_n} (1 - A^2)^n$$

$$= (n + 1)(1 - A^2)^n \xrightarrow{n \to \infty} 0$$

so $\{p_n\}_{n \geq 1}$ is an approximate unit in $\mathcal{E}$. 
4. Let $f$ be a function in $\mathcal{E}$ that vanishes outside of $[-\frac{1}{2}, \frac{1}{2}]$. Prove that, for every $n \geq 1$, the function $p_n \ast f$ is polynomial on its support. 

First observe that $p_n(x - t)$ is a polynomial in $x$. To fix notations, we write

$$p_n(x - t) = \sum_{k=0}^{2n} c_k(t)x^k.$$ 

Then, for $x$ in the support of the convolution,

$$(f \ast p_n)(x) = \sum_{k=0}^{2n} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)c_k(t) \, dt \right)x^k$$

which is a polynomial expression.

5. **Prove Weierstrass’ Theorem.**

It follows from the previous results that a continuous function with compact support in $[-\frac{1}{2}, \frac{1}{2}]$ is a uniform limit of polynomial functions. Now let $f$ be a continuous function defined on a segment $[a,b]$. Extend $f$ to a function $\tilde{f} \in \mathcal{E}$. This can be done for instance by requesting that $\tilde{f}$ be 0 outside of $[a - 1, b + 1]$, coincide with $f$ on $[a,b]$ and affine elsewhere.

An affine transformation from $[a - 1, b + 1]$ to $[-\frac{1}{2}, \frac{1}{2}]$ allows to use the result proved in 4. and to conclude.