Problem 1 (Distance to a subset and metric Urysohn’s Lemma).
Let \((E, d)\) be a metric space. For any subset \(A \subset E\) and any point \(x \in E\), the \textit{distance} between \(x\) and \(A\) is defined by

\[
d(x, A) = \inf_{a \in A} d(x, a).
\]

1. Show that \(d(x, A) = d(x, \overline{A})\).

2. Show that \(d(\cdot, A)\) is 1-Lipschitz.

3. Let \(A\) and \(B\) be disjoint closed subsets of \(E\). Prove the existence of a continuous function \(f : E \to \mathbb{R}\) such that:

(a) \(0 \leq f(x) \leq 1\) for all \(x \in E\);
(b) \(f(x) = 0\) for all \(x \in A\);
(c) \(f(x) = 1\) for all \(x \in B\).

\textbf{Solution.} 1. Observe that \(A \subset \overline{A}\) so \(d(x, A) \geq d(x, \overline{A})\). For the other inequality, consider \(\alpha\) in \(\overline{A}\). There exists a sequence \(\{a_n\} \in A^\mathbb{N}\) that converges to \(\alpha\). Given \(x\) fixed, the function \(d(x, \cdot)\) is continuous so \(\lim_{n \to \infty} d(x, a_n) = d(x, \alpha)\).

Since \(d(x, a_n) \geq d(x, A)\) for every \(n\), it follows that \(d(x, \alpha) \geq d(x, A)\). This is true for every \(\alpha\) in \(\overline{A}\) so \(d(x, \overline{A}) \geq d(x, A)\).

2. For \(x, y \in E\) and \(a \in A\), the triangle inequality and the definition of \(d(x, A)\) imply that \(d(x, A) \leq d(x, y) + d(y, a)\). This is true for every \(a \in A\) so \(d(x, A) \leq d(x, y) + d(y, A)\) and we get \(d(x, A) - d(y, A) \leq d(x, y)\). The same argument gives \(d(y, A) - d(x, A) \leq d(x, y)\) hence the result.
3. Consider \( x \mapsto \frac{d(x, A)}{d(x, A) + d(x, B)} \).

\[
\text{Problem 2 (Completeness is not a topological property).}
\]

Let \( E = (0, +\infty) \) and for \( x, y \in E \), consider \( \delta(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right| \).

1. Prove that \( \delta \) is a distance on \( E \) and that it induces the same topology as the Euclidean distance \( d \).

2. Is the map \( x \mapsto x^{-1} \) uniformly continuous as a map from \((E, d)\) to itself? As a map from \((E, d)\) to \((E, \delta)\)?

3. Is \((E, \delta)\) complete? What about \(([0, 1], d)\) and \(([0, 1], \delta)\)?

**Solution.**

1. **Method 1:** prove that every open \( d \)-ball contains a \( \delta \)-ball with the same center and vice versa. **Method 2:** prove that \((E, d) \xrightarrow{\text{Id}} (E, \delta)\) is a homeomorphism. To see this, it is convenient to decompose the identity map as \((E, d) \xrightarrow{\varphi} (E, d) \xrightarrow{\varphi} (E, \delta)\) where \( \varphi(x) = x^{-1} \) and prove that both are homeomorphisms. Note that both methods boil down to the fact that \( \varphi \) is a homeomorphism from \((E, d \text{ or } \delta)\) to \((E, d \text{ or } \delta)\).

2. No. Yes.

3. No: \( u_n = n \) is Cauchy but it does not converge (argue by contradiction). No: it is not closed in \((\mathbb{R}, d)\). Alternatively, consider \( u_n = \frac{1}{n} \), Cauchy but not convergent in \((0, 1]\).

Yes. **Method 1:** show that a Cauchy sequence \( \{u_n\} \) for \( \delta \) is also Cauchy for \( d \) hence converges for \( d \) in the closure of \((0, 1]\). If the \( d \)-limit is \( > 0 \), it is also the \( \delta \)-limit because \( d \) and \( \delta \) induce the same topology (or check it directly with balls) so the sequence converges. Assume the limit is \( 0 \). Then \( \delta(1, u_n) \) diverges to \(+\infty\) so \( \{u_n\} \) is not bounded which is impossible since it is Cauchy. **Method 2:** \( x \mapsto x^{-1} \) is an isometry (hence uniformly continuous) between \(([0, 1], \delta)\) and \(([1, +\infty), d)\), which is closed in \((\mathbb{R}, d)\) complete, so is complete.

\[\blacksquare\]
Problem 3 (The Banach Contraction Principle).
Let \((E, d)\) be a complete metric space and \(f : E \longrightarrow E\).

1. Show that if \(f\) is \(k\)-Lipschitz with \(k < 1\), the equation \(f(x) = x\) has a unique solution in \(E\).

2. Show that if \(E\) is compact, it is enough to have \(d(f(x), f(y)) < d(x, y)\) for all \(x, y\) to obtain the same result.

Solution. 1. Think triangle inequality and geometric series.

2. The real-valued function \(x \mapsto d(x, f(x))\) is continuous on a compact set so it is bounded and the extrema are attained.

\[ \square \]

Problem 4 (Completeness of \(\ell^2(\mathbb{N})\)).
Show that the set of sequences \(U = \{u_n\}\) such that \(\sum_{n \geq 0} |u_n|^2\) converges is complete for the norm \(\|U\|_2 = \left( \sum_{n=0}^{\infty} |u_n|^2 \right)^{\frac{1}{2}}\).

Solution. The skeleton of the proof we studied for the space of bounded functions with values in a complete space carries over.

\[ \square \]

Problem 5 (Cantor’s Intersection Theorem).
Let \((E, d)\) be a metric space and \(A \subset E\) a non-empty subset. The diameter of \(A\) is defined by
\[ \text{diam}(A) = \sup_{x, y \in A} d(x, y). \]

Prove that \(E\) is complete if and only if for every decreasing sequence \(\{F_n\}_{n \in \mathbb{N}}\) of closed subsets of \(E\) such that \(\lim_{n \to \infty} \text{diam}(F_n) = 0\), there is a point \(x\) such that
\[ \bigcap_{n \in \mathbb{N}} F_n = \{x\}. \]

Solution. See Section 9.4 of [Royden-Fitzpatrick].
Problem 6 (Characterizations of compactness for metric spaces).
Let \((E, d)\) be a metric space. Prove that the following conditions are equivalent.

(i) \(E\) has the Borel-Lebesgue property, i.e. is topologically compact.

(ii) If \(\mathcal{F}\) is a family of closed subsets of \(E\) such that every subfamily has nonempty intersection, then \(\bigcap_{F \in \mathcal{F}} F \neq \emptyset\).

(iii) \(E\) is complete and totally bounded i.e. can be covered by finitely many open balls of radius \(\varepsilon\), for any \(\varepsilon > 0\).

(iv) \(E\) has the Bolzano-Weierstrass property, i.e. is sequentially compact.

Solution. The equivalence between (i) and (ii) holds in topological (non-necessarily metric) spaces. See Propositions 17, 18 and 19 in Section 9.5 of [Royden-Fitzpatrick] for the rest.