APPLICATIONS OF THE ARZELÀ-ASCOLI
AND THE BAIRE CATEGORY THEOREMS

MATH 113 - SPRING 2015

PROBLEM SET #2

Problem 1 (Hölder maps).
A function $f \in C([0, 1], \mathbb{R})$ is said to be $\alpha$-Hölder if

$$h_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite. For $M > 0$ and $0 < \alpha \leq 1$, denote

$$H_{\alpha, M} = \{ f \in C([0, 1], \mathbb{R}) , h_\alpha(f) \leq M \text{ and } \|f\|_\infty \leq M \} .$$

Prove that $H_{\alpha, M}$ is compact in $(C([0, 1], \mathbb{R}), \| \cdot \|_\infty)$.

Solution. The Arzelà-Ascoli Theorem implies that it suffices to check that $H_{\alpha, M}$ is closed, bounded and equicontinuous. The set in question is the intersection of the closed ball $B_c(0, M)$ and $F = \{ f \in C([0, 1]), h_\alpha(f) \leq M \}$, so it is automatically bounded and it is enough to check that $F$ is closed. To do so, consider a sequence $\{f_n\}$ of functions in $F$, that converges to $f$ in $C([0, 1])$. The pointwise convergence of the sequence implies that $\frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq M$ for every $x \neq y$ so $F$ is closed. To establish equicontinuity, let $\varepsilon > 0$ and verify that $\delta = \left( \frac{\varepsilon}{M} \right)^\frac{1}{\alpha}$ is an appropriate modulus of continuity.
**Problem 2.** Show that a normed linear space over $\mathbb{R}$ that has a countable algebraic basis cannot be complete.

**Solution.** Let $E$ be a normed space with an algebraic basis $\{e_i\}_{i \in \mathbb{N}}$ and $F_n = \text{span}(e_1, \ldots, e_n)$. Each $F_n$ is finite-dimensional, hence closed. Moreover, if $F_n$ contained an open ball of radius $r > 0$ it would also contain $B(0, r)$, which generates $E$, so $E$ would be contained in $F_n$. Therefore, each $F_n$ has empty interior and Baire’s Theorem ensures that $\bigcup_{n \geq 1} F_n$ has empty interior too, which contradicts the fact that $\bigcup_{n \geq 1} F_n = E$. 

**Problem 3.** Let $f : (0, +\infty) \to \mathbb{R}$ be continuous and assume that for all $x > 0$,

$$\lim_{n \to \infty} f(nx) = 0.$$ 

Prove that $\lim_{x \to \infty} f(x) = 0$.

**Hint:** for $\varepsilon > 0$ and $n \in \mathbb{N}$, consider $F_{n, \varepsilon} = \{x \geq 0, \forall p \geq n, |f(px)| \leq \varepsilon\}$.

**Solution.** Each $F_{n, \varepsilon}$ is closed as the intersection of inverse images of the closed subset $[0, \varepsilon]$ of $\mathbb{R}$ by the continuous functions $f(p \cdot)$ for $p \in \mathbb{N}$, $p \geq n$. The hypothesis on $f$ implies that $(0, +\infty) \subset \bigcup_{n \geq 1} F_n$. Being locally compact, $(0, +\infty)$ is a Baire space so that there exists $n_0 \in \mathbb{N}$ such that $F_{n_0} \neq \emptyset$.

In other words, there exist $0 < \alpha < \beta$ such that $(\alpha, \beta) \subset F_{n_0}$, which means that

$$\forall x \in (\alpha, \beta) , \forall p \geq n_0 , |f(px)| \leq \varepsilon.$$ 

The result then follows from the fact that, for $p$ large enough, the intervals $(p \alpha, p \beta)$ overlap. More precisely, the condition $(p + 1)\alpha < p\beta$ is equivalent to $p > \frac{\alpha}{\beta - \alpha}$ so that if $N > \max(n_0, \frac{\alpha}{\beta - \alpha})$, one has $|f(x)| \leq \varepsilon$ for $x$ in $\bigcup_{p \geq N} (p\alpha, p\beta) = (N \alpha, +\infty)$. 

$\square$
Problem 4. Show that nowhere differentiable functions are dense in $E = C([0,1], \mathbb{R})$ equipped with its ordinary norm.

Hint: consider, for $\varepsilon > 0$ and $n \in \mathbb{N},$

$U_{n,\varepsilon} = \left\{ f \in E : \forall x \in [0,1] \text{, } \exists y \in [0,1] \text{, } |x - y| < \varepsilon \text{ and } \left| \frac{f(y) - f(x)}{y - x} \right| > n \right\}.$

Solution. We first prove that each set $U_{n,\varepsilon}$ is open because its complement $U_{n,\varepsilon}^c$ is closed. Observe that

$U_{n,\varepsilon}^c = \left\{ f \in E : \exists x \in [0,1] \text{, } \forall y \in [0,1] \text{, } |x - y| < \varepsilon \Rightarrow \left| \frac{f(y) - f(x)}{y - x} \right| \leq n \right\}.$

and let $\{f_k\}$ be a sequence in $U_{n,\varepsilon}^c$ that converges to $f$ in $E$. For each $k$, there exists $x_k \in [0,1]$ such that $|x_k - y| < \varepsilon \Rightarrow \left| \frac{f(y) - f(x_k)}{y - x_k} \right| \leq n$. Since $[0,1]$ is compact, $\{x_k\}$ has a convergent subsequence $\{x_{\varphi(k)}\}$. Denote $x$ its limit and let $y$ in $[0,1]$ be such that $0 < |x - y| < \varepsilon$. For $k$ large enough, one has $0 < |x_{\varphi(k)} - y| < \varepsilon$ so that $\left| \frac{f_{\varphi(k)}(y) - f_{\varphi(k)}(x_{\varphi(k)})}{y - x_{\varphi(k)}} \right| \leq n$ and the uniform convergence $f_{\varphi(k)} \to f$ implies that $\left| \frac{f(y) - f(x)}{y - x} \right| \leq n$, so that $f$ belongs to $U_{n,\varepsilon}^c$.

Now we prove that $U_{n,\varepsilon}$ is dense in $E$. Polynomials are dense in $E$, so it suffices to prove that functions of class $C^1$ can be approximated by elements of $U_{n,\varepsilon}$.

For $p \geq 1$ integer, let $v_p$ be a continuous function on $[0,1]$, affine on each interval $\left[ \frac{k}{2p}, \frac{k+1}{2p} \right]$ and such that $v_p \left( \frac{k}{2p} \right) = 0$ (resp. $= 1$) if $k$ is even (resp. odd). Let $f$ be a function of class $C^1$ on $[0,1]$ and $g_p = f + \lambda v_p$. By construction, $\|f - g_p\|_\infty \leq \lambda$ so $g_p$ can be chosen arbitrarily close to $f$.

If $x \neq y$ in $[0,1]$, then

$\left| \frac{g_p(x) - g_p(y)}{x - y} \right| \geq \lambda \left| \frac{v_p(x) - v_p(y)}{x - y} \right| - \left| \frac{f(x) - f(y)}{x - y} \right| \geq \lambda \left| \frac{v_p(x) - v_p(y)}{x - y} \right| - ||f'||_\infty.$

Let $p > \frac{1}{2\lambda} (n + ||f'||_\infty)$. For any $x \in [0,1]$, there exists $y \in [0,1]$ within $\varepsilon$ of $x$ and in the same interval $\left[ \frac{k}{2p}, \frac{k+1}{2p} \right]$. By definition of $v_p$, the latter implies that
\[ \frac{|v_p(x) - v_p(y)|}{x - y} = 2p. \] Then

\[ \left| \frac{g_p(x) - g_p(y)}{x - y} \right| \geq 2p\lambda - \|f'\|_\infty > n \]

so that \( g_p \in U_{n,\varepsilon} \).

The Baire Category Theorem ensures that \( U = \bigcap_{n \geq 1} U_{1/n, n} \) is dense in \( E \). Let \( f \in U \) and \( x \in [0, 1] \). Then there is a sequence \( \{x_n\} \) such that \( 0 < |x_n - x| < \frac{1}{n} \) and

\[ \left| \frac{f(x_n) - f(y)}{x_n - y} \right| > n, \] which prevents \( f \) from being differentiable at \( x \). 

\[ \square \]