Problem 1 (Gram-Schmidt orthonormalization). Let $\mathcal{X} = \{x_n\}_{n \geq 0}$ be a countable family of linearly independent vectors in a Hilbert space. Prove the existence of a countable orthonormal family $\mathcal{Y} = \{y_n\}_{n \geq 0}$ such that

$$\text{Span}(x_0, \ldots, x_p) = \text{Span}(y_0, \ldots, y_p)$$

for all $p \geq 0$.

Problem 2 (Orthogonal polynomials). Let $I$ be an interval of $\mathbb{R}$ and $w : I \to \mathbb{R}$ a continuous positive function such that $x \mapsto x^n w(x)$ is integrable on $I$ for any integer $n \geq 0$. Denote by $C$ the set of continuous functions $f : I \to \mathbb{R}$ such that $x \mapsto f^2(x) w(x)$ is integrable. Finally, for $f$ and $g$ real-valued functions on $I$, we define

$$\langle f, g \rangle_w = \int_I f(x) g(x) w(x) \, dx$$

1. Verify that $\mathbb{R}[X] \subset C$ and that $\langle \cdot, \cdot \rangle_w$ is an inner product on $C$. Denote by $\| \cdot \|_w$ the corresponding norm. Is $(C, \| \cdot \|_w)$ a Hilbert space?

2. Prove the existence of an orthonormal basis $\{P_n\}_{n \geq 0}$ of $\mathbb{R}[X]$ such that the degree of $P_n$ is $n$ and its leading coefficient $\gamma_n$ is positive.

3. Verify that the polynomials $P_n$ satisfy a relation of the form

$$P_n = (a_n X + b_n)P_{n-1} + c_n P_{n-2}$$

and determine the sequences $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ and $\{c_n\}_{n \in \mathbb{N}}$.

4. Prove that $P_n$ has $n$ distinct roots in $I$. 
5. Assume \( I \) compact.
   (a) Find a constant \( C \) such that \( \|f\|_w \leq C\|f\|_\infty \) for all \( f \in \mathcal{C} \).
   (b) For \( f \) in \( \mathcal{C} \), let \( p_n(f) \) be the orthogonal projection of \( f \) on \( \mathbb{R}_n[X] \). Prove that \( p_n(f) \xrightarrow{n \to \infty} f \).

Hint: 1. You may choose a concrete \( w \) to study completeness. 2. Project \((\dagger)\) and express \( a_n \) in terms of \( \gamma_n \) and \( \gamma_{n-1} \). 4. Compute \( \langle P_n, \prod_\alpha (X - \alpha) \rangle_w \) where the product is taken over roots of \( P_n \) with odd order.

**Problem 3.** Let \( G \) be a group acting on a countable set \( X \). Let \( \mathcal{H} = l^2(X) \) be the Hilbert space of square-integrable functions on \( X \) for the counting measure.

1. Let \( A \) and \( B \) be subsets of \( X \), with indicators denoted by \( \chi_A \) and \( \chi_B \).
   (a) Give a condition on \( A \), equivalent to \( \chi_A \in \mathcal{H} \).
   (b) Give a condition on \( A \) and \( B \), equivalent to \( \chi_A \perp \chi_B \) in \( \mathcal{H} \).

2. For \( f \in \mathcal{H} \) and \( g \in G \), define \( \pi(g)f = x \mapsto f(g^{-1} \cdot x) \).
   (a) Prove that each \( \pi(g) \) is a unitary operator on \( \mathcal{H} \).
   (b) Prove that \( \pi : G \rightarrow U(\mathcal{H}) \) is a group homomorphism.

From now on, we assume that for every \( x \in X \), the \( G \)-orbit \( \{g \cdot x \mid g \in G\} \) is infinite.

3. Let \( A \subset X \) be such that \( \chi_A \in \mathcal{H} \) and denote by \( \mathcal{C} \) be the closure of the convex hull\(^1\) of \( \mathcal{C}_0 = \{\pi(g)\chi_A \mid g \in G\} \).
   (a) Prove the existence of a unique element \( \xi \) of minimal norm in \( \mathcal{C} \).
   (b) Verify that \( \mathcal{C} \) is stable by each of the operators \( \pi(g) \).
   (c) Prove that \( \pi(g)\xi = \xi \) for all \( g \in G \).
   (d) Deduce that \( \xi \) is constant on each \( G \)-orbit and conclude.

4. Let \( A, B \) be non-empty finite subsets of \( X \) and assume that \( (g \cdot A) \cap B \neq \emptyset \) for all \( g \) in \( G \).
   (a) Prove that \( \langle f, \chi_B \rangle \geq 1 \) for all \( f \in \mathcal{C} \).
   (b) Apply the previous result to \( \xi \) and conclude.

\(^1\)the convex hull of a set \( S \) is the family of all possible convex combinations of elements of \( S \).