(PRE-)HILBERT SPACES

MATH 113 - SPRING 2015

PROBLEM SET #7

Problem 1 (Gram-Schmidt orthonormalization)**.** Let $\mathcal{X} = \{x_n\}_{n\geq 0}$ be a countable family of linearly independent vectors in a Hilbert space. Prove the existence of a countable orthonormal family $\mathcal{Y} = \{y_n\}_{n\geq 0}$ such that

$$\mathrm{Span}(x_0, \dots, x_p) = \mathrm{Span}(y_0, \dots, y_p)$$

for all $p \geq 0$.

Solution. Let $y_0 = \dfrac{1}{\|x_0\|} x_0$. Assume constructed $y_0, \dots, y_n$ satisfying the requirements. The projection of x_{n+1} on $\mathrm{Span}(y_0, \dots, y_n)$ is $\sum_{k=0}^{n} \langle x_{n+1}, y_k \rangle \, y_k$ so

$$y'_{n+1} = x_{n+1} - \sum_{k=0}^{n} \langle x_{n+1}, y_k \rangle \, y_k$$

is orthogonal to all the vectors y_k for $k \leq n$. Switching y'_{n+1} and x_{n+1} accross the equality symbol and the induction hypothesis show the equality of the generated subspaces, and it suffices to define $y_{n+1} = \dfrac{1}{\|y'_{n+1}\|} y'_{n+1}$. □

Problem 2 (Orthogonal polynomials)**.** Let I be an interval of $\mathbb{R}$ and $w : I \to \mathbb{R}$ a continuous positive function such that $x \mapsto x^n w(x)$ is integrable on I for any integer $n \geq 0$. Denote by $\mathcal{C}$ the set of continuous functions $f : I \to \mathbb{R}$ such that $x \mapsto f^2(x)w(x)$ is integrable. Finally, for f and g real-valued functions on I, we define

$$\langle f, g \rangle_w = \int_I f(x) g(x) w(x) \, dx$$

1. Verify that $\mathbb{R}[X] \subset \mathcal{C}$ and that $\langle \cdot, \cdot \rangle_w$ is an inner product on $\mathcal{C}$. Denote by $\| \cdot \|_w$ the corresponding norm. Is $(\mathcal{C}, \| \cdot \|_w)$ a Hilbert space?

2. Prove the existence of an orthonormal basis $\{P_n\}_{n\geq 0}$ of $\mathbb{R}[X]$ such that the degree of P_n is n and its leading coefficient γ_n is positive.

3. Verify that the polynomials P_n satisfy a relation of the form

$$P_n = (a_n X + b_n)P_{n-1} + c_n P_{n-2} \qquad (\dagger)$$

and determine the sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$.

4. Prove that P_n has n distinct roots in I.

5. Assume I compact.

   (a) Find a constant C such that $\|f\|_w \leq C\|f\|_\infty$ for all $f \in \mathcal{C}$.

   (b) For f in $\mathcal{C}$, let $p_n(f)$ be the orthogonal projection of f on $\mathbb{R}_n[X]$. Prove that $p_n(f) \xrightarrow[n\to\infty]{\|\cdot\|_w} f$.

Hint: 1. You may choose a concrete w to study completeness. 3. Project (†) and express a_n in terms of γ_n and γ_{n-1}. 4. Compute $\langle P_n, \prod_\alpha (X - \alpha)\rangle_w$ where the product is taken over roots of P_n with odd order.

Solution. 1. Bilinearity comes from properties of the integral, positivity and definiteness result from the assumptions on the weight w. However, $(\mathcal{C}, \| \cdot \|_w)$ is not complete as discontinuous functions with finite $\| \cdot \|_w$ norm can be obtained as limits of Cauchy sequences in $\mathcal{C}$.

2. Apply the Gram-Schmidt procedure to the canonical basis of $\mathbb{R}[X]$ and multiply by -1 if necessary to guarantee that $\gamma_n > 0$.

3. First assume that such a relation exists. Then, projecting onto the lines generated by P_n, P_{n-1} and P_{n-2} leads to

$$1 = a_n\langle XP_{n-1}, P_n\rangle_w \quad 0 = a_n\langle XP_{n-1}, P_{n-1}\rangle_w + b_n \quad 0 = a_n\langle XP_{n-1}, P_{n-2}\rangle_w + c_n.$$

A direct computation shows that $a_n = \dfrac{\gamma_n}{\gamma_{n-1}}$. Similarly, $b_n = -\dfrac{\gamma_n}{\gamma_{n-1}}\langle XP_{n-1}, P_{n-1}\rangle$ and $c_n = -\dfrac{\gamma_{n-2}}{\gamma_{n-1}^2}$. Choosing these values for a_n, b_n and c_n guarantees that

$P_n - ((a_n X + b_n)P_{n-1} + c_n P_{n-2})$ has degree at most $n-3$. This polynomial is a combination of P_n, P_{n-2} and XP_{n-1}. The first two are orthogonal to any P_k with $k \leq n-3$ by construction. For the last one, observe that $\langle XP_{n-1}, P_k\rangle = \langle P_{n-1}, XP_k\rangle = 0$ since $XP_k \in \mathrm{span}(P_0, \ldots, P_{n-2})$. This implies that $P_n - ((a_n X + b_n)P_{n-1} + c_n P_{n-2}) = 0$.

4. Let $Q = \prod_\alpha (X - \alpha)$ where α runs over the roots odd order of P_n, with the convention that $Q = 1$ if there are no such roots. If Q has degree $< n$, then $P_n \perp Q$ by definition of the family $\{P_n\}$. On the other hand the function $x \mapsto P_n(x)Q(x)w(x)$ is non-negative so its integral is 0 only if it is constantly 0, which it is not. Therefore Q has degree n and P_n has n distinct roots in I.

5. (a) A direct estimate gives $C = \sqrt{\int_I w}$. (b) Let $\varepsilon > 0$. By Stone-Weierstrass there exists a polynomial S such that $\|f - S\|_\infty < \frac{\varepsilon}{C}$. Let N be its degree. By optimality of the orthogonal projection, $\|f - p_N(f)\|_w \leq \|f - S\|_w \leq C\|f - S\|_\infty < \varepsilon$. Bessel's Inequality implies that $\{\|f - p_n(f)\|_w\}$ is a decreasing sequence and the result follows.

Note: families of orthogonal polynomials for various weights have many applications in a variety of contexts. In the case of $I = (-1, 1)$ with $w(x) = (1 - x^2)^{-\frac{1}{2}}$, one obtains the **Chebyshev polynomials of the first kind**. They are subject to the relation $P_n = 2xP_{n-1} - P_{n-2}$ and satisfy the relation $P_n(cos\theta) = \cos n\theta$. They are very useful in Approximation Theory. **Legendre polynomials** correspond to the case of $I = [-1, 1]$ with $w(x) = 1$, **Hermite polynomials** to the case of $I = \mathbb{R}$ with $w(x) = e^{-x^2}$ and **Laguerre polynomials** to the case of $I = [0, \infty)$ with $w(x) = e^{-x}$. □

Problem 3. Let G be a group acting on a countable set X. Let $\mathcal{H} = \ell^2(X)$ be the Hilbert space of square-integrable functions on X for the counting measure.

1. Let A and B be subsets of X, with indicators denoted by χ_A and χ_B.
   (a) Give a condition on A, equivalent to $\chi_A \in \mathcal{H}$.
   (b) Give a condition on A and B, equivalent to $\chi_A \perp \chi_B$ in $\mathcal{H}$.

2. For $f \in \mathcal{H}$ and $g \in G$, define $\pi(g)f = x \mapsto f(g^{-1} \cdot x)$.
   (a) Prove that each $\pi(g)$ is a unitary operator on $\mathcal{H}$.

(b) Prove that $\pi : G \longrightarrow \mathrm{U}(\mathcal{H})$ is a group homomorphism.

From now on, we assume that for every $x \in X$, the G-orbit $\{g \cdot x \ , \ g \in G\}$ is infinite.

3. Let $A \subset X$ be such that $\chi_A \in \mathcal{H}$ and denote by C be the closure of the convex hull[1] of $C_0 = \{\pi(g)\chi_A \ , \ g \in G\}$.
   (a) Prove the existence of a unique element ξ of minimal norm in C.
   (b) Verify that C is stable by each of the operators $\pi(g)$.
   (c) Prove that $\pi(g)\xi = \xi$ for all $g \in G$.
   (d) Deduce that ξ is constant on each G-orbit and conclude.
4. Let A, B be non-empty finite subsets of X and assume that $(g \cdot A) \cap B \neq \varnothing$ for all g in G.
   (a) Prove that $\langle f, \chi_B \rangle \geq 1$ for all $f \in C$.
   (b) Apply the previous result to ξ and conclude.

Solution. 1. Observe that $\langle \chi_A, \chi_B \rangle = \mathrm{Card}\, A \cap B$ so that $\chi_A \in \mathcal{H}$ if and only if A is finite and $\chi_A \perp \chi_B$ if and only if A and B are disjoint.

2. Each map $x \mapsto g \cdot x$ is a bijection so the sums defining $\|f\|_2^2$ and $\|\pi(g)f\|_2^2$ only differ by the order of the terms. The morphism property follows from the fact that $(gh)^{-1} = h^{-1}g^{-1}$ in any group.

3. (a) The set C is convex as the closure of a convex set and ξ is the projection of 0 on C.
   (b) By construction, C_0 is stable by each $\pi(g)$. These maps are continuous so C is stable too.
   (c) Since $\pi(g)$ is an isometry, $\|\pi(g)\xi\| = \|\xi\|$. Now ξ is the only element in C with norm $\|\xi\|$ so $\pi(g)\xi = \xi$.
   (d) We have $\xi(g^{-1} \cdot x) = \pi(g)\xi(x) = \xi(x)$ for all $x \in X$ and $g \in G$ so ξ is constant on the orbits. The only constant square-integrable function on an infinite discrete space is 0 so $\xi = 0$.

[1] the convex hull of a set S is the family of all possible convex combinations of elements of S.

4. The hypothesis on A and B implies that $\langle f, \chi_B \rangle \geq 1$ for all f of the form $\pi(g)\chi_A$. It extends to f in C_0 by convex combinations and to all of C by continuity of the inner product. In particular, we should have $\langle \xi, \chi_B \rangle \geq 1$, which contradicts the fact that $\xi = 0$.

□