Problem 1 (Gram-Schmidt orthonormalization). Let $\mathcal{X} = \{x_n\}_{n \geq 0}$ be a countable family of linearly independent vectors in a Hilbert space. Prove the existence of a countable orthonormal family $\mathcal{Y} = \{y_n\}_{n \geq 0}$ such that

$$\text{Span}(x_0, \ldots, x_p) = \text{Span}(y_0, \ldots, y_p)$$

for all $p \geq 0$.

Solution. Let $y_0 = \frac{1}{\|x_0\|} x_0$. Assume constructed $y_0, \ldots, y_n$ satisfying the requirements. The projection of $x_{n+1}$ on $\text{Span}(y_0, \ldots, y_n)$ is $\sum_{k=0}^{n} \langle x_{n+1}, y_k \rangle y_k$ so

$$y'_{n+1} = x_{n+1} - \sum_{k=0}^{n} \langle x_{n+1}, y_k \rangle y_k$$

is orthogonal to all the vectors $y_k$ for $k \leq n$. Switching $y'_{n+1}$ and $x_{n+1}$ across the equality symbol and the induction hypothesis show the equality of the generated subspaces, and it suffices to define $y_{n+1} = \frac{1}{\|y'_{n+1}\|} y'_{n+1}$. \hfill \square

Problem 2 (Orthogonal polynomials). Let $I$ be an interval of $\mathbb{R}$ and $w : I \to \mathbb{R}$ a continuous positive function such that $x \mapsto x^n w(x)$ is integrable on $I$ for any integer $n \geq 0$. Denote by $C$ the set of continuous functions $f : I \to \mathbb{R}$ such that $x \mapsto f^2(x) w(x)$ is integrable. Finally, for $f$ and $g$ real-valued functions on $I$, we define

$$\langle f, g \rangle_w = \int_I f(x) g(x) w(x) \, dx$$
1. Verify that $\mathbb{R}[X] \subset C$ and that $\langle \cdot, \cdot \rangle_w$ is an inner product on $C$. Denote by $\| \cdot \|_w$ the corresponding norm. Is $(C, \| \cdot \|_w)$ a Hilbert space?

2. Prove the existence of an orthonormal basis $\{P_n\}_{n \geq 0}$ of $\mathbb{R}[X]$ such that the degree of $P_n$ is $n$ and its leading coefficient $\gamma_n$ is positive.

3. Verify that the polynomials $P_n$ satisfy a relation of the form

$$P_n = (a_nX + b_n)P_{n-1} + c_nP_{n-2} \quad (\dagger)$$

and determine the sequences $\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}$ and $\{c_n\}_{n \in \mathbb{N}}$.

4. Prove that $P_n$ has $n$ distinct roots in $I$.

5. Assume $I$ compact.

   (a) Find a constant $C$ such that $\|f\|_w \leq C\|f\|_\infty$ for all $f \in C$.

   (b) For $f$ in $C$, let $p_n(f)$ be the orthogonal projection of $f$ on $\mathbb{R}_n[X]$. Prove that $p_n(f) \xrightarrow{\|\cdot\|_n} f$.

**Hint:** 1. You may choose a concrete $w$ to study completeness. 3. Project $(\dagger)$ and express $a_n$ in terms of $\gamma_n$ and $\gamma_{n-1}$. 4. Compute $\langle P_n, \prod_\alpha (X - \alpha) \rangle_w$ where the product is taken over roots of $P_n$ with odd order.

**Solution.**

1. Bilinearity comes from properties of the integral, positivity and definiteness result from the assumptions on the weight $w$. However, $(C, \| \cdot \|_w)$ is not complete as discontinuous functions with finite $\| \cdot \|_w$ norm can be obtained as limits of Cauchy sequences in $C$.

2. Apply the Gram-Schmidt procedure to the canonical basis of $\mathbb{R}[X]$ and multiply by $-1$ if necessary to guarantee that $\gamma_n > 0$.

3. First assume that such a relation exists. Then, projecting onto the lines generated by $P_n, P_{n-1}$ and $P_{n-2}$ leads to

$$1 = a_n\langle XP_{n-1}, P_n \rangle_w \quad 0 = a_n\langle XP_{n-1}, P_{n-1} \rangle_w + b_n \quad 0 = a_n\langle XP_{n-1}, P_{n-2} \rangle_w + c_n.$$ 

A direct computation shows that $a_n = \frac{\gamma_n}{\gamma_{n-1}}$. Similarly, $b_n = -\frac{\gamma_n}{\gamma_{n-1}}\langle XP_{n-1}, P_{n-1} \rangle$ and $c_n = -\frac{\gamma_{n-2}}{\gamma_{n-1}}$. Choosing these values for $a_n, b_n$ and $c_n$ guarantees that...
\[ P_n - ((a_nX + b_n)P_{n-1} + c_nP_{n-2}) \] has degree at most \( n - 3 \). This polynomial is a combination of \( P_n, P_{n-2} \) and \( XP_{n-1} \). The first two are orthogonal to any \( P_k \) with \( k \leq n - 3 \) by construction. For the last one, observe that \( \langle XP_{n-1}, P_k \rangle = \langle P_{n-1}, XP_k \rangle = 0 \) since \( XP_k \in \text{span}(P_0, \ldots, P_{n-2}) \). This implies that \( P_n - ((a_nX + b_n)P_{n-1} + c_nP_{n-2}) = 0 \).

4. Let \( Q = \prod_\alpha (X - \alpha) \) where \( \alpha \) runs over the roots odd order of \( P_n \), with the convention that \( Q = 1 \) if there are no such roots. If \( Q \) has degree \( n \), then \( P_n \perp Q \) by definition of the family \( \{P_n\} \). On the other hand the function \( x \mapsto P_n(x)Q(x)w(x) \) is non-negative so its integral is 0 only if it is constantly 0, which it is not. Therefore \( Q \) has degree \( n \) and \( P_n \) has \( n \) distinct roots in \( I \).

5. (a) A direct estimate gives \( C = \sqrt{\int_I w} \). (b) Let \( \varepsilon > 0 \). By Stone-Weierstrass there exists a polynomial \( S \) such that \( \|f - S\|_\infty < \frac{\varepsilon}{C} \). Let \( N \) be its degree. By optimality of the orthogonal projection, \( \|f - p_N(f)\|_w \leq \|f - S\|_w \leq C\|f - S\|_\infty < \varepsilon \). Bessel’s Inequality implies that \( \{\|f - p_n(f)\|_w \} \) is a decreasing sequence and the result follows.

**Note:** families of orthogonal polynomials for various weights have many applications in a variety of contexts. In the case of \( I = (-1, 1) \) with \( w(x) = (1 - x^2)^{-\frac{1}{2}} \), one obtains the **Chebyshev polynomials of the first kind**. They are subject to the relation \( P_n = 2xP_{n-1} - P_{n-2} \) and satisfy the relation \( P_n(\cos \theta) = \cos n\theta \). They are very useful in Approximation Theory. **Legendre polynomials** correspond to the case of \( I = [-1, 1] \) with \( w(x) = 1 \), **Hermite polynomials** to the case of \( I = \mathbb{R} \) with \( w(x) = e^{-x^2} \) and **Laguerre polynomials** to the case of \( I = [0, \infty) \) with \( w(x) = e^{-x^2} \).

**Problem 3.** Let \( G \) be a group acting on a countable set \( X \). Let \( \mathcal{H} = \ell^2(X) \) be the Hilbert space of square-integrable functions on \( X \) for the counting measure.

1. Let \( A \) and \( B \) be subsets of \( X \), with indicators denoted by \( \chi_A \) and \( \chi_B \).
   
   (a) Give a condition on \( A \), equivalent to \( \chi_A \in \mathcal{H} \).
   
   (b) Give a condition on \( A \) and \( B \), equivalent to \( \chi_A \perp \chi_B \) in \( \mathcal{H} \).

2. For \( f \in \mathcal{H} \) and \( g \in G \), define \( \pi(g)f = x \mapsto f(g^{-1} \cdot x) \).
   
   (a) Prove that each \( \pi(g) \) is a unitary operator on \( \mathcal{H} \).
(b) Prove that $\pi : G \longrightarrow U(\mathcal{H})$ is a group homomorphism.

From now on, we assume that for every $x \in X$, the $G$-orbit $\{g \cdot x, \ g \in G\}$ is infinite.

3. Let $A \subset X$ be such that $\chi_A \in \mathcal{H}$ and denote by $C$ be the closure of the convex hull\(^1\) of $C_0 = \{\pi(g)\chi_A, \ g \in G\}$.

(a) Prove the existence of a unique element $\xi$ of minimal norm in $C$.

(b) Verify that $C$ is stable by each of the operators $\pi(g)$.

(c) Prove that $\pi(g)\xi = \xi$ for all $g \in G$.

(d) Deduce that $\xi$ is constant on each $G$-orbit and conclude.

4. Let $A, B$ be non-empty finite subsets of $X$ and assume that $(g \cdot A) \cap B \neq \emptyset$ for all $g$ in $G$.

(a) Prove that $\langle f, \chi_B \rangle \geq 1$ for all $f \in C$.

(b) Apply the previous result to $\xi$ and conclude.

**Solution.**

1. Observe that $\langle \chi_A, \chi_B \rangle = \text{Card} \ A \cap B$ so that $\chi_A \in \mathcal{H}$ if and only if $A$ is finite and $\chi_A \perp \chi_B$ if and only if $A$ and $B$ are disjoint.

2. Each map $x \mapsto g \cdot x$ is a bijection so the sums defining $\|f\|_2^2$ and $\|\pi(g)f\|_2^2$ only differ by the order of the terms. The morphism property follows from the fact that $(gh)^{-1} = h^{-1}g^{-1}$ in any group.

3. (a) The set $C$ is convex as the closure of a convex set and $\xi$ is the projection of 0 on $C$.

(b) By construction, $C_0$ is stable by each $\pi(g)$. These maps are continuous so $C$ is stable too.

(c) Since $\pi(g)$ is an isometry, $\|\pi(g)\xi\| = \|\xi\|$. Now $\xi$ is the only element in $C$ with norm $\|\xi\|$ so $\pi(g)\xi = \xi$.

(d) We have $\xi(g^{-1} \cdot x) = \pi(g)\xi(x) = \xi(x)$ for all $x \in X$ and $g \in G$ so $\xi$ is constant on the orbits. The only constant square-integrable function on an infinite discrete space is 0 so $\xi = 0$.

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\(^1\)the convex hull of a set $S$ is the family of all possible convex combinations of elements of $S$. 

4. The hypothesis on $A$ and $B$ implies that $\langle f, \chi_B \rangle \geq 1$ for all $f$ of the form $\pi(g)\chi_A$. It extends to $f$ in $C_0$ by convex combinations and to all of $C$ by continuity of the inner product. In particular, we should have $\langle \xi, \chi_B \rangle \geq 1$, which contradicts the fact that $\xi = 0$. 

$\square$