Problem 1 (Pointwise and uniform convergence). Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a $2\pi$-periodic function, piecewise continuous, piecewise of class $C^1$. For $x_0 \in \mathbb{R}$, we denote by $f(x_0^\pm)$ the one-sided limit $\lim_{x \rightarrow x_0^\pm} f(x)$ and $\tilde{f}$ is the function defined on $\mathbb{R}$ by

$$\tilde{f}(x) = \frac{f(x^+) + f(x^-)}{2}.$$ 

The purpose of the problem is to establish the pointwise convergence of the Fourier series of $f$ to $\tilde{f}$, that is, for any $x_0 \in \mathbb{R}$,

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx_0} = \tilde{f}(x_0).$$

1. Verify that for any $x_0 \in \mathbb{R}$, the map $h \mapsto \frac{f(x_0 + h) + f(x_0 - h) - f(x_0^+) - f(x_0^-)}{h}$ is bounded near 0.

First, we consider the case $x_0 = 0$. Denote by $S_N(f)(0)$ the partial sum $\sum_{n=-N}^{N} \hat{f}(n)$.

2. Prove that

$$2\pi S_N(f)(0) = \int_{0}^{\pi} (f(x) + f(-x)) D_N(x) \, dx,$$

where $D_N(x)$ is the Dirichlet kernel $\frac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}$.

3. Show that $2\pi (S_N(f)(0) - \tilde{f}(0))$ can be written as $\int_{0}^{\pi} g(x) \sin \left( N + \frac{1}{2} \right) x \, dx$ with $g$ piecewise continuous and bounded near 0.
4. Conclude and extend to the case of arbitrary $x_0$.

From now on, we assume $f$ continuous and piecewise of class $C^1$. We denote by $\varphi$ the function defined on $\mathbb{R}$ by

$$\varphi(x) = \begin{cases} 
  f'(x) & \text{if } f \text{ is differentiable at } x, \\
  \frac{f'(x^+) + f'(x^-)}{2} & \text{otherwise}.
\end{cases}$$

5. Verify the relation $\hat{\varphi}(n) = in \hat{f}(n)$ for all $n \in \mathbb{Z}$.

6. Prove that the Fourier series of $f$ converges normally to $f$.

**Hints:** 4. Riemann-Lebesgue. Consider $f_{x_0} : x \mapsto f(x + x_0)$. 6. $|ab| \leq \frac{1}{2}(a^2 + b^2)$.

**Problem 2** (Application to the computation of sums). Let $f$ be the $2\pi$-periodic function on $\mathbb{R}$ defined by $f(x) = 1 - \frac{x^2}{\pi^2}$ for all $x \in [-\pi, \pi]$.

1. Compute the Fourier coefficients of $f$.

2. Deduce the sums of the series $\sum_{n \geq 1} \frac{1}{n^2}$, $\sum_{n \geq 1} \frac{(-1)^n}{n^2}$ and $\sum_{n \geq 1} \frac{1}{n^4}$.

**Hints:** note that only the real part of $\hat{f}(n)$ is useful. Parseval.

**Problem 3** (Not every function is equal to the sum of its Fourier series). Let $C_{2\pi}$ denote the space of $2\pi$-periodic continuous functions on $\mathbb{R}$, equipped with $\| \cdot \|_\infty$.

For $N \in \mathbb{N}$, we define a linear functional $\varphi_N$ on $C_{2\pi}$ by

$$\varphi_N(f) = S_N(f)(0) = \sum_{n=-N}^{N} \hat{f}(n).$$

1. Verify that $C_{2\pi}$ is a Banach space.

2. Prove that $\varphi_N \in C_{2\pi}^*$ and compute $\|\varphi_N\|$.

3. Show that $\|\varphi_N\| \geq \frac{2}{\pi} \int_0^{(2N+1)\pi} \left| \frac{\sin u}{u} \right| du$ for any $N \in \mathbb{N}$.

4. Prove the existence of a function in $C_{2\pi}$ whose Fourier series diverges at 0.

**Hints:** 2. Consider $f_\varepsilon = \frac{D_N}{|D_N| + \varepsilon}$. 4. Use the Principle of Uniform Boundedness.