Here is a simple proof of the existence of lots of continuous nowhere differentiable functions on the real line. The argument here follows the outline given in Pedersen's *Analysis Now* [1]. In fact, I will prove the following theorem.

Theorem 1. *The collection of continuous nowhere differentiable functions is dense in the Banach space $X = C\big([0,1]\big)$ of continuous functions on $[0,1]$ with the supremum norm.*

The first step is to consider the collection $\mathscr{F}_n$ of $f \in X$ with the property that there is a $x_f \in [0,1]$ such that $\big|f(y) - f(x_f)\big| \le n|y - x_f|$ for all $y \in [0,1]$.

Lemma 2. *For each $n \ge 1$, $\mathscr{F}_n$ is closed in X.*

Proof. Suppose that $\{ f_k \} \subseteq \mathscr{F}_n$ and converges to f in X. For notational convenience, I'll write x_k for x_{f_k}. Using the compactness of $[0,1]$, we can, by passing to a subsequence and relabeling, assume that $\{ x_k \}$ converges to $x \in [0,1]$. Since $f_k \to f$ *uniformly*, $\{ f_k(x_k) \}$ converges to $f(x)$. Therefore for all $y \in [0,1]$,

$$\begin{aligned} \big|f(y) - f(x)\big| &= \lim_{k\to\infty} \big|f_k(y) - f_k(x_k)\big| \\ &\le n \lim_{k\to\infty} |y - x_k| = n|y - x|. \end{aligned}$$

That is, f belongs to $\mathscr{F}_n$. □

Lemma 3. *If $f \in X$ and if f is differentiable at $x \in [0,1]$, then $f \in \bigcup_{n=1}^{\infty} \mathscr{F}_n$.*

Proof. It is straightforward to see that there is a $\delta > 0$ so that $|y - x| < \delta$ implies that

$$\big|f(y) - f(x)\big| \le \big(\big|f'(x)\big| + 1\big)|y - x|.$$

Thus f is in $\mathscr{F}_n$ for any $n \ge \max\big\{2\delta^{-1}\|f\|_\infty, \big|f'(x)\big| + 1\big\}$. □

In the sequel, it will be important to remember that a continuous, piecewise linear function always has one-sided derivatives at every point. I'll use the notation

$$D^+f(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad D^-f(x) = \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h}.$$

If $\big|D^+f(x)\big| \ge n$ and $\big|D^-f(x)\big| \ge n$ for all $x \in [0,1]$, then I'll write $f \in \mathrm{PW}_n$. Thus PW_n is the collection of continuous, piecewise linear functions whose one-sided derivatives are *always* numerically larger than n. It will also be handy to let ϕ be the continuous function on $\mathbf{R}$ of period one determined by

$$\phi(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \quad \text{and} \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Finally let $\phi_n(x) = 2^{-n}\phi(4^n x)$, and notice that ϕ_n is in PW_{2^n} and satisfies $\|\phi_n\|_\infty \le 2^{-n}$. Now we can make our final observation.

Lemma 4. *If $f \in X$, $\epsilon > 0$, and $N \in \mathbf{Z}^+$, then there is a $g \in \mathrm{PW}_N$ such that $\|f - g\|_\infty < \epsilon$.*

Proof. Since f is uniformly continuous, there is a $m \in \mathbf{Z}^+$ such that $|x - y| < 1/m$ implies that $\bigl|f(x) - f(y)\bigr| < \epsilon/2$. Let $x_i = i/m$ for $i = 0, 1, \dots, m$, and define

$$g_0\bigl(\lambda x_i + (1 - \lambda)x_{i+1}\bigr) = lf(x_i) + (1 - \lambda)f(x_{i+1})$$

for $i = 0, 1, \dots, m - 1$ and $0 \le \lambda \le 1$. Then g_0 is a continuous, piecewise linear function on $[0, 1]$ which satisfies $\|f - g_0\|_\infty < \epsilon/2$. Let $M = \max_{0 \le i \le m-1} m\bigl|f(x_{i+1}) - f(x_i)\bigr|$. Then $\bigl|D^+ g_0(x)\bigr| \le M$ for all $x \in [0, 1]$ (and similarly for $\bigl|D^- g_0(x)\bigr|$). Thus if we take k such that $2^k \ge M + N$ and $2^{-k} < \epsilon/2$, then $g = g_0 + \phi_k$ will satisfy the requirements of the lemma. □

Proof of Theorem 1. Lemmas 2 and 4 imply that each $\mathscr{F}_n$ is closed with empty interior in X. Therefore each $\mathscr{O}_n = \mathscr{F}_n^c$ is open and dense. The Baire Category Theorem then implies that

$$\left(\bigcup_{n=1}^{\infty} \mathscr{F}_n\right)^c = \bigcap_{n=1}^{\infty} \mathscr{O}_n$$

is dense in X. The theorem now follows from Lemma 4. □

Remark 5. We've actually shown that the collection of nowhere differentiable functions are a bit more than dense in $C\bigl([0, 1]\bigr)$. In a complete metric space X, the countable intersection of dense open sets must be of "second category;" in particular, such a set must be uncountable if X is.

References

[1] Gert K. Pedersen, *Analysis now*, Graduate Texts in Mathematics, vol. 118, Springer-Verlag, New York, 1989. MR90f:46001