Extra Material

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The product toplogy is one of the more ubiquitous objects in elementary topology. Let (X_a, τ_a) be a topological space for all $a \in A$. Recall that the Cartesian product $\prod_{a\in A} X_a$ is the set of all functions $x : A \to \bigcup_{a\in A} X_a$ such that $x(a) \in X_a$. If $a_0 \in A$, then the projection p_{a_0} onto the a_0-factor is the map $p_{a_0} : \prod_{a\in A} X_a \to X_{a_0}$ given by $p_{a_0}(x) = x(a_0)$. The product topology on $\prod_{a\in A} X_a$ is the initial topology induced by the projections maps. Thus the product topology is the smallest topology on the product such that each projection is continuous. A subbasis is given by the sets $U(a, V) = p_a^{-1}(V)$ for any $a \in A$ with V open in X_a.

1. Let (x_λ) be a net in $Z = \prod_{a\in A} X_a$. Then $x_\lambda \to x$ in the product topology if and only if $x_\lambda(a) \to x(a)$ for all $a \in A$. (So the product topology can be thought of as the topology of pointwise convergence.)

ANS: Suppose that $x_\lambda \to x$ in Z and $a \in A$. Let V be a neighborhood of $x(a)$. Since (x_λ) is eventually in $U(a, V) = p_a^{-1}(V)$, $(x_\lambda(a))$ is eventually in V. This proves $x_\lambda(a) \to x(a)$ as required.

Now suppose that $x_\lambda(a) \to x(a)$ for all a. A basic open neigborhood of x is of the form

$$U = U(a_1, V_1) \cap \cdots \cap U(a_n, V_n).$$

But there is an λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda(a_k) \in V_k$ for $k = 1, \ldots, n$. But then $\lambda \geq \lambda_0$ implies $x_\lambda \in U$. This suffices.

The **Tychonoff Theorem** asserts that the (arbitrary) product of compact spaces is compact in the product topology. We'll use this to prove the **Alaoglu Theorem** in due course. Right now, I want to point out that #229 does not hold in general topological spaces.

2. For each $\alpha \in \ell^\infty$, let D_α be a closed disk in $\mathbf{C}$ such that $\alpha_n \in D_\alpha$ for all $n \geq 1$. Then $Z = \prod_{\alpha\in\ell^\infty} D_\alpha$ is compact in the product topology. Let $(z_n) \subset Z$ be the sequence given by $z_n(\alpha) = \alpha_n$. Then (z_n) has accumulation points (just because Z is compact and applying #228), but no converent subsequences.

ANS: Let (x_n) be as above. Suppose to the contrary, (x_n) has a subsequence (x_{n_k}) converging to x. Then $x_{n_k}(\alpha) \to x(\alpha)$ for all $\alpha \in \ell^\infty$. Define $\alpha_0 \in \ell^\infty$ as follows:

$$\alpha_0(n) = \begin{cases} (-1)^k & \text{if } n = n_k \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

But then we get a contradiction since $x_{n_k}(\alpha_0) = (-1)^k$ which does not converge to anything.

The proof the Tychonoff Theorem is one of those things that is often omitted in standard courses. The excuse is time (as in this course) and often the proof is unsatisfactory in that it uses machinery beyond the scope of the current course. Fortunately, Zach Garvey pointed me to a tidy proof in Loomis's "Abstract Harmonic Analysis" [2]. Here it is in it's entirety.

Theorem (Tychonoff)**.** *Suppose that X_α is compact for all $\alpha \in A$. Then $X = \prod_{\alpha \in A} X_\alpha$ is compact in the product topology.*

Proof. Let $\mathcal{F}$ be a family of closed sets in X with the FIP (finite intersection property). We need to prove that

$$\bigcap_{F \in \mathcal{F}} F \neq \emptyset$$

By Zorn's Lemma, there is a maximal family of (not necessarily closed) subsets $\mathcal{F}_0$ such that $\mathcal{F} \subset \mathcal{F}_0$ and such that $\mathcal{F}_0$ has the FIP. Notice that the maximality condition on $\mathcal{F}_0$ implies that $\mathcal{F}_0$ is closed under intersection.

For each $\alpha \in A$, let $\mathcal{F}_0^\alpha$ be the collection of subsets of X_α which are images of elements of $\mathcal{F}_0$ under the projection map $p_\alpha : X \to X_\alpha$. Note that $\mathcal{F}_0^\alpha$ has the FIP for each α. Since X_α is compact, there is a point $x_\alpha \in X_\alpha$ that belongs to the closure of each element in $\mathcal{F}_0^\alpha$.

Let $x \in X$ be given by $x(\alpha) = x_\alpha$. It will suffice to see that $x \in \overline{F}$ for all $F \in \mathcal{F}_0$; that implies $x \in F$ for all $F \in \mathcal{F}$.

Let U be a neighborhood of x in X. Then

$$x \in \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i}) \subset U$$

for $\alpha_1, \dots, \alpha_n \in A$ and open sets $U_{\alpha_i} \subset X_{\alpha_i}$. Thus $x_{\alpha_i} \in U_{\alpha_i}$ and U_{α_i} meets every set in $\mathcal{F}_0^{\alpha_i}$. Hence $p_{\alpha_i}^{-1}(U_{\alpha_i})$ meets every set in $\mathcal{F}_0$. By maximality and the fact that $\mathcal{F}_0$ is closed under intersection, $p_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}_0$. But then $\bigcap_{i=1}^n p_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}_0$. Similarly, $U \in \mathcal{F}_0$.

This means that U meets each $F \in \mathcal{F}_0$. Since U is an arbitrary neighborhood of x, $x \in \overline{F}$ for all $F \in \mathcal{F}_0$. This completes the proof. □

The Jordan-von Neumann Theorem

This proof was taken from [1, Chap. XII §7, Exercises 19–24]. It is not really necessary to assume that X is complete and a similar proof works over $\mathbf{R}$.

Proposition 1 (Jordan-von Neumann Theorem)**.** *Suppose that X is a complex*[1] *Banach space whose norm satisfies the parallelogram law:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Then X is a Hilbert space. More precisely, the form

$$(x \mid y) := \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2 \tag{1}$$

is an inner product on X such that $(x \mid x) = \|x\|^2$.

We proceed with a sequence of lemmas.

Lemma 1. *For each $x \in X$, $(x \mid x) = \|x\|^2$.*

Proof. Using the homogeneity of $\|\cdot\|$:

$$4(x \mid x) = \|2x\|^2 + i\|(1+i)x\|^2 - 0 - i\|(1-i)x\|^2 = 4\|x\|^2. \quad \square$$

Corollary 1. *For all $x \in X$, $(x \mid x) \geq 0$ and $(x \mid x) = 0$ only if $x = 0$.*

Lemma 2. *For all $x, y \in X$, we have $(y \mid x) = \overline{(x \mid y)}$.*

Proof. Again, using the homogeneity of $\|\cdot\|$:

$$\begin{aligned} 4(x \mid y) &= \|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2 \\ &= \|x+y\|^2 + i\|y-ix\|^2 - \|y-x\|^2 - i\|y+ix\|^2 \\ &= 4\overline{(y \mid x)} \quad \square \end{aligned}$$

The next lemma is the key step. Of course, it was suggested by the exercise in Knapp's book. Nevertheless, it still found it tricky to work out. I am confident that there is a better way.

Lemma 3. *For all $x, y, z \in X$, we have*

$$\|x+y+z\|^2 = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2.$$

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[1] This proof can be easily modified for a real Banach space: simply replace (1) with

$$(x \mid y) := \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big).$$

Proof. We make repeated use of the parallelogram law (to the indicated term):

$$\begin{aligned}
\|x+y+z\|^2 &= 2\|x+y\|^2 + 2\|z\|^2 - \|x+y-z\|^2 \\
&= \|x+y\|^2 + \|z\|^2 + \underbrace{\|x+y\|^2 + \|z\|^2} - \|x+y-z\|^2 \\
&= \|x+y\|^2 + \|z\|^2 + \frac{1}{2}\underbrace{\|x+y+z\|^2} - \frac{1}{2}\|x+y-z\|^2 \\
&= \|x+y\|^2 + \|z\|^2 + \frac{1}{2}\big(2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2\big) - \frac{1}{2}\|x+y-z\|^2 \\
&= \|x+y\|^2 + \|x+z\|^2 + \|y\|^2 + \|z\|^2 - \frac{1}{2}\underbrace{\big(\|x-y+z\|^2 + \|x+y-z\|^2\big)} \\
&= \|x+y\|^2 + \|x+z\|^2 + \underbrace{\|y\|^2 + \|z\|^2} - \|x\|^2 - \|z-y\|^2 \\
&= \|x+y\|^2 + \|x+z\|^2 + \frac{1}{2}\|y+z\|^2 + \frac{1}{2}\|z-y\|^2 - \|x\|^2 - \|z-y\|^2 \\
&= \|x+y\|^2 + \|x+z\|^2 + \frac{1}{2}\|y+z\|^2 - \|x\|^2 - \frac{1}{2}\underbrace{\|z-y\|^2} \\
&= \|x+y\|^2 + \|x+z\|^2 + \frac{1}{2}\|y+z\|^2 - \|x\|^2 - \frac{1}{2}\big(2\|y\|^2 + 2\|z\|^2 - \|y+z\|^2\big) \\
&= \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2. \quad \square
\end{aligned}$$

Having successfully dealt with that messy computation, the fact that the potential inner product preserves sums is easy.

Lemma 4. *For all $x, y, z \in X$, we have $(x+y \mid x) = (x \mid z) + (y \mid z)$.*

Proof. The essential observation is that $\sum_{k=0}^{3} i^k = 1 + i - 1 - i = 0$. Then using Lemma 3, we have

$$\begin{aligned}
4(x+y \mid x) &= \sum_{k=0}^{3} i^k \|x+y+i^k z\|^2 \\
&= \sum_{k=0}^{3} i^k \big(\|x+y\|^2 + \|x+i^k z\|^2 + \|y+i^k z\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2\big) \\
&= 0 + \sum_{k=0}^{3} \|x+i^k z\|^2 + \sum_{k=0}^{3} \|y+i^k z\|^2 + 0 + 0 + 0 \\
&= 4(x \mid z) + 4(y \mid z). \quad \square
\end{aligned}$$

It would seem now that we are all but done. But showing that the potential inner product respects scalar multiplication is not so easy. We have to work with the complex rationals $\mathbf{D} = \mathbf{Q} + i\mathbf{Q}$.

Lemma 5. *Suppose that $r \in \mathbf{D}$ and that $x, y \in X$, then $(rx \mid y) = r(x \mid y)$.*

Proof. It follows immediately from Lemma 4, that for all $n \in \mathbf{N}$, we have $(nx \mid y) = n(x \mid y)$. It is then a simple matter to see that $(rx \mid y) = r(x \mid y)$ for all $r \in \mathbf{Q}$. On the other hand,

$$\begin{aligned} 4(ix \mid y) &= \sum_{k=0}^{3} i^k \|ix + i^k y\|^2 \\ &= \|ix + y\|^2 + i\|ix + iy\|^2 - \|ix - y\|^2 - i\|ix - iy\|^2 \end{aligned}$$

which, by homogeneity, is

$$\begin{aligned} &= i\big(-i\|x - iy\|^2 + \|x + y\|^2 - i\|x - iy\|^2 - \|x - y\|^2\big) \\ &= 4i(x \mid y). \end{aligned}$$

combining this with the first part of the proof and Lemma 4 gives the result. □

Upgrading from $r \in \mathbf{Q}$ to $c \in \mathbf{C}$ requires that we prove that the Cauchy Schwarz inequality holds using only the tools at our disposal so far. Fortunately, the usual proofs works just fine.

Lemma 6. *For all $x, y \in X$, $|(x \mid y)| \le \|x\|\|y\|$.*

Proof. We can assume that $y \neq 0$. Note that for all $r \in \mathbf{D}$,

$$\begin{aligned} 0 \le \|x - ry\|^2 &= (x - ry \mid x - ry) \\ &= \|x\|^2 - 2\operatorname{Re}\bar{r}(x \mid y) + |r|^2\|y\|^2. \end{aligned}$$

But we can find a sequence or rationals $r_n \to \frac{(x|y)}{\|y\|^2}$. Then taking limits in the above,

$$0 \le \|x\|^2 - 2\frac{|(x \mid y)|}{\|y\|^2} + \frac{|(x \mid y)|}{\|y\|^2},$$

and the result follows. □

Lemma 7. *For all $c \in \mathbf{C}$, we have $(cx \mid y) = c(x \mid y)$.*

Proof. Let $r_n \to c$ with each $r_n \in \mathbf{D}$. Then using Lemma 6, we have

$$|(r_n x \mid y) - (cx \mid y)| \leq |r_n - c| \|x\| \|y\|,$$

and $(r_n x \mid y) \to (cx \mid y)$. But by Lemma 5, $(r_n x \mid y) = r_n(x \mid y) \to c(x \mid y)$. This suffices. □

Proof of Proposition 1. It follows from Lemma 4 and Lemma 7 that $(\cdot \mid \cdot)$ is linear in its first variable. By Lemma 2 it is conjugate linear in its second variable, and both positive and definite by Corollary 1. Therefore $(\cdot \mid \cdot)$ is an inner product. It defines the original, and therefore complete, norm on X by Lemma 1. The Proposition follows. □

References

[1] Anthony W. Knapp, *Basic real analysis*, Cornerstones, Birkhäuser Boston Inc., Boston, MA, 2005, Along with a companion volume *Advanced real analysis*. MR MR2155259 (2006c:26002)

[2] Lynn H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.