TAKE HOME MIDTERM
DUE THURSDAY, FEBRUARY 23

MATH 113, WINTER 06, INSTRUCTOR: MARIUS IONESCU

There are 10 problems for this midterm. Good luck and remeber that I root for you!

(1) Problem 9 on page 23.

(2) Let $\mathcal{H}$ be a Hilbert space.If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ let $P_{\mathcal{M}}$ be the projection onto $\mathcal{M}$.
   (a) $\mathcal{M}_1 \subset \mathcal{M}_2$ if and only if $P_{\mathcal{M}_1} P_{\mathcal{M}_2} = P_{\mathcal{M}_1} \iff P_{\mathcal{M}_2} P_{\mathcal{M}_1} = P_{\mathcal{M}_1}$.
   (b) $\mathcal{M}_1 = \mathcal{M}_2$ if and only if $P_{\mathcal{M}_1} = P_{\mathcal{M}_2}$.
   (c) $\mathcal{M}_1 \perp \mathcal{M}_2$ if and only $P_{\mathcal{M}_1} P_{\mathcal{M}_2} = 0$.
   (d) If $\mathcal{M}_1, \dots, \mathcal{M}_n$ are closed subspaces of $\mathcal{H}$, then

$$\mathcal{H} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n \iff P_{\mathcal{M}_1} + \cdots P_{\mathcal{M}_n} = I.$$

(3) Problem 4 on page 30.

(4) We say that two operators S and T in $\mathcal{B}(\mathcal{H})$ are unitarily equivalent if there is a unitary U such that $S = UTU*$.
   (a) Show that unitary equivalence is an equivalence relation.
   (b) Show that unitary equivalence preserves norm, self-adjointness, normality and unitarity.

(5) Suppose that P and Q are projections in $\mathcal{B}(\mathcal{H})$ which are unitarily equivalent via U (see the previous problem). Find an explicit unitary V in $\mathrm{Mat}_2(\mathcal{B}(\mathcal{H}))$ in terms of U such that

$$V \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} V^* = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}.$$

(6) If $\mathcal{H}$ is a complex Hilbert space and $A \in \mathcal{B}(\mathcal{H})$, we define the numerical radius of A as

$$|||A||| = \sup\{|\langle Ax, x\rangle| \ : \ x \in \mathcal{H}, \|x\| \leq 1\}.$$

Show that $|||\cdot|||$ is a norm on $\mathcal{B}(\mathcal{H})$ which satisfies

$$\frac{1}{2}\|A\| \leq |||A||| \leq \|A\|, \quad A \in \mathcal{B}(\mathcal{H}).$$

(Hint: you can use the polarization identity).

(7) Let $\mathcal{H} = \mathbb{C}^2$ and $E \in \mathcal{B}(\mathcal{H})$ be an idempotent. Let $P = P_{E\mathcal{H}}$ be the orthogonal projection onto the range of E. Prove that

$$P = (EE^* + (1-E)^*(1-E))^{-1}EE^* = (1 + (E^* - E)^*(E^* - E))^{-1}EE^*.$$

(8) For two projections P and Q in $\mathcal{B}(\mathcal{H})$ we define $P \leq Q$ if $P\mathcal{H} \subset Q\mathcal{H}$. We say that P and Q are perpendicular $P \perp Q$ if $P\mathcal{H} \perp Q\mathcal{H}$. We define also

$$P \wedge Q = P_{P\mathcal{H} \cap Q\mathcal{H}} \ , \ P \vee Q = P_{\overline{P\mathcal{H} + Q\mathcal{H}}}.$$

Show that
   (a) $P \leq Q$ if and only if $PQ = P = QP$.

(b) $P \perp Q \iff PQ = 0 \iff P \leq 1 - Q$.
(c) $P \wedge Q$ is the greatest projection $E \in \mathcal{B}(\mathcal{H})$ with $E \leq P, E \leq Q$.
(d) $P \vee Q$ is the smallest projection $E \in \mathcal{B}(\mathcal{H})$ with $E \geq P, E \geq Q$.

(9) Problem 3 on page 45.
(10) Problem 6 on pge 45.