Optional Assignment on Nets
Please do NOT turn in.

1. Suppose that X is a first countable space. Show that each $x \in X$ has a neighborhood basis of open sets $\{ U_n \}_{n=1}^{\infty}$ such that $U_{n+1} \subseteq U_n$.

2. Let X and Y be 1^{st} countable spaces.

(a) Show that $\mathcal{O} \subseteq X$ is open in X if and only if every *sequence* converging to some $x \in \mathcal{O}$ is eventually in $\mathcal{O}$.

(b) Show that $F \subseteq X$ is closed if and only if every convergent *sequence* in F converges to a point in F.

(c) Show that $f : X \to Y$ is continuous if and only if whenever $\{x_n\}_{n=1}^{\infty}$ coverges to $x \in X$ then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in Y$.

(d)* Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Show that if $\{x_n\}_{n=1}^{\infty}$ has a convergent *subnet*, then $\{x_n\}_{n=1}^{\infty}$ has a convergent *subsequence*. (Hint: if $\{x_n\}_{n=1}^{\infty}$ has an accumulation point, then it must have a convergent subsequence.)

(e)* Show that 1^{st} countability is required in part (d). (Hint: let ℓ^{∞} denote the set of *bounded* sequences. If $\alpha = \{\alpha_n\}_{n=1}^{\infty} \in \ell^{\infty}$, let I_{α} be any closed bounded interval in $\mathbf{R}$ such that $\alpha_n \in I_{\alpha}$ for all n. Set

$$Z = \prod_{\alpha \in \ell^{\infty}} I_{\alpha}.$$

Consider the sequence $\{x_n\}_{n=1}^{\infty}$ in the compact space Z defined by $x_n(\alpha) = \alpha_n$.)