Homework Assignment #5
Due Friday March 9th.

1. Suppose that $P \in B(\mathcal{H})$ is a projection (that is, $P = P^* = P^2$). Show that $P(\mathcal{H})$ is closed and that P is the orthogonal projection onto $W := P(\mathcal{H})$

2. Let $\{ e_j \}_{j \in J}$ be an orthonormal basis for $\mathcal{H}$. Show that U is unitary if and only if $\{ Ue_j \}_{j \in J}$ is an orthonormal basis for $\mathcal{H}$.

3. Let $\Theta_{x,y}$ be the rank-one operator $\Theta_{x,y}(z) := (z \mid y)x$. Suppose that $T \in B(H)$.

(a) Show that $T\Theta_{x,y} = \Theta_{Tx,y}$ and $\Theta_{x,y}T = \Theta_{x,T^*y}$.

(b) Show that $\big(\Theta_{x,y}\big)^* = \Theta_{y,x}$.

(c) Show that $\|\Theta_{x,y}\| = \|x\|\|y\|$.

(d) Show that $T \in B_f(H)$ if and only if $T(H)$ is finite dimensional.

(e) Show that if $T \in B_f(H)$, then $T^* \in B_f(H)$, and that if $T \in B_f(H)$ and $S \in B(H)$, then both TS and ST are in $B_f(H)$.

4. Suppose that P and Q are projections in $B(H)$. We say that $P \perp Q$ if $P(H) \perp Q(H)$ and that $P \leq Q$ if $P(H) \subset Q(H)$.

(a) Show that the following are equivalent.

(i) $P \perp Q$.

(ii) $PQ = QP = 0$.

(iii) $P + Q$ is a projection.

(b) Show that the following are equivalent.

(i) $P \leq Q$.

(ii) $PQ = QP = P$.

(iii) $Q - P$ is a projection.

(Hint: Note that PQP is a positive operator. Also $PQP = PQ(PQ)^*$ so that $PQP = 0$ if and only if $PQ = QP = 0$.)

5. Work E 3.3.1 in the text.

6. Work E 3.3.2 in the text.

7. Work E 3.3.4 in the text.

8. Recall that we can identify the dual H^* of a Hilbert space H with itself via the conjugate linear isomorphism $\Phi : H \to H^*$ given by $\Phi(x)(y) := (y \mid x)$. Then we have two definitions for the adjoint: $T^* : H \to H$ as in Theorem 3.2.3 and $T^* : H^* \to H^*$ as in 2.3.9. How do they compare. What does Proposition 2.4.12 say about adjoints *a la* Chapter 3? Are any of these statements hard to prove directly?