Math 74/114, Spring 2017

Homework set 4, due Wed May 3

This homework set is due on Wednesday May 3, at the start of class. Discussion of the problems is permitted, and even recommended. But you should write up and hand in your own solutions.

1. Do Hatcher's Exercises 4, 5, 7, 9 on p. 79.

2. Let $F_2 = \langle a, b \rangle$ be the free group on two generators.

   (a) If $\varphi : F_2 \to \mathbb{Z}/2\mathbb{Z}$ is the homomorphism determined by $\varphi(a) = 0$, $\varphi(b) = 1$, prove that the kernel of φ is the subgroup of F_2 generated by elements $a, b^2, bab^{-1} \in F_2$. (*Hint. Characterize the reduced words in F_2 that are in the kernel of φ, and show that they can all be written as products of a, b^2, bab^{-1}.*)

   (b) Prove that the kernel of φ is a free group. Look at figure (1) on p. 58 in Hatcher.

   (c) Prove that F_2 has a subgroup that is isomorphic to F_n, the free group on n generators, for any $n = 1, 2, 3, \ldots$.

   (d) Prove the Nielsen-Schreier theorem: All subgroups of a free group F_n are free groups.

3. A topological group G is a group that is also a topological space, while multiplication and inverses are continuous maps,

$$m : G \times G \to G,\ m(g_1, g_2) = g_1 g_2$$
$$i : G \to G,\ i(g) = g^{-1}$$

   Assume that G is path-connected, locally path-connected, and semilocally simply connected. Think, for example, of Lie groups like $\mathrm{SO}(n)$, $\mathrm{U}(n)$, etc., which are manifolds.

   (a) Define a continuous group operation $\tilde{G} \times \tilde{G} \to \tilde{G}$ and continuous inverse map $\tilde{G} \to \tilde{G}$ for the universal cover $\tilde{G}$ of G, with the property that the covering map $p : \tilde{G} \to G$ is a group homomorphism. Verify that the group axioms hold.

   (b) Prove that the group structure of $\tilde{G}$ is *uniquely* determined by the property that the covering map is a homomorphism.

   (c) Show that $\tilde{G}$ contains a normal subgroup N that is isomorphic to $\pi_1(G)$, and that $G \cong \tilde{G}/N$.