# Main Results (so far)

Let $F$ be a field of characteristic not 2, and $(V, B)$ a quadratic space over $F$. Some important results:

- (I.1.3 Lam) If $V$ is regular, and $U$ any subspace, then
  1. $\dim V = \dim U + \dim U^\perp$.
  2. $(U^\perp)^\perp = U$.
- (I.2.3 Lam) (Representation Criterion) Let $(V, B)$ be a quadratic space and $d \in \dot{F}$. Then $d \in D(V)$ if and only if there exists another quadratic space $(V', B')$ together with an isometry $V \cong \langle d \rangle \perp V'$.
- (I.2.4 Lam) Orthogonal diagonalization of $(V, B)$.
- (I.2.5 Lam) If $(V, B)$ is a quadratic space (not necessarily regular), and $U$ is a regular subspace, then
  1. $V = U \perp U^\perp$.
  2. If $W$ is a subspace with $V = U \perp W$, then $W = U^\perp$.
- (I.2.6 Lam) If $V$ is regular, then a subspace $U \subseteq V$ is regular if and only if there exists a subspace $W$ of $V$ with $V = U \perp W$.

  **Remarks:**
  1. There are easy counterexamples if $V$ is not regular (split the radical).
  2. We can prove the following proposition: If $U$ is a subspace of a quadratic space $V$ (not necessarily regular), then if $V = U \perp U^\perp$, then $U$ is regular.

     This gets around $V$ being regular in I.2.6, at the expense of specifying $W = U^\perp$.

     *Proof:* $\operatorname{rad}(U) = \operatorname{rad}_U(U) \subset U \cap U^\perp = \{0\}$.
- Any representation $\varphi : V \to W$ with $V$ regular is injective.
- (42.8 O'Meara) Let $V$ and $W$ be quadratic spaces with radical splittings, $V = V^\perp \perp V_1$ and $W = W^\perp \perp W_1$. Then
  1. There is a representation $V \to W$ if and only if there is a representation $V_1 \to W_1$.
  2. $V \cong W$ if and only if $V^\perp \cong W^\perp$ and $V_1 \cong W_1$.

- (3.2 Lam) Let $V$ be a binary quadratic space over a field $F$. The following are equivalent:
  1. $V$ is a hyperbolic plane, i.e., $V \cong \mathbb{H} \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
  2. $V$ is isotropic and regular.
  3. $dV = -1$.
- (2.25 Gerstein)
  1. A hyperplane plane is universal.
  2. If $V$ is regular and isotropic, then it is split by a hyperbolic plane, that is $V \cong H \perp V'$ where $H \cong \mathbb{H} \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- (2.43. Gerstein, 3.4 Lam) Let $V$ be a regular quadratic space and $U \subseteq V$ a totally isotropic subspace with basis $\{u_1, \ldots, u_r\}$. Then there is an orthogonal splitting of $V$: $V \cong H_1 \perp \cdots \perp H_r \perp V'$ with $H_i \cong \mathbb{H}$ and $u_i \in H_i$. In particular, $\dim U \leq \frac{1}{2} \dim V$.
- (2.27 Gerstein, 3.5 Lam) [The Representation Theorem] Let $V$ be a regular quadratic space, and $\alpha \in \dot{F}$. Then $\alpha \to V$ if and only if $V \perp \langle -\alpha \rangle$ is isotropic.