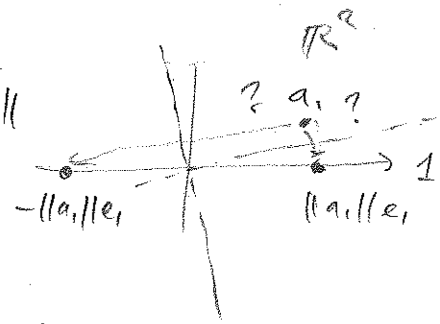
[stat Lec 6]. ③ 4/8/14.

finishing (WS) $\to$ $H = I - 2\frac{vv^T}{\|v\|^2}$

$v = a_1 \pm e_1\|a_1\|$




two choices of plane $v$.

pick sign?

weight $v^T a_1 = [a_{11} \pm \|a_1\|, f^T]\begin{bmatrix} a_{11} \\ f \end{bmatrix}$

$= \underbrace{a_{11}^2 \pm \|a_1\| a_{11}}_{a_{11}(a_{11} \pm \|a_1\|)} + \underbrace{\|f\|^2}_{}$ say small, as in fig

want add, not cancel (C.C.).

so $v = a_1 + \text{sgn}(a_{11}) \|a_1\| e_1$ $\to$ is which in fig? long one.

(counter intuitive not short one.)

Remaining cols?

$Q_1 A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$ — now work in $\mathbb{R}^{m-1}$ & find $\tilde{Q}_2 \in \mathbb{R}^{(m-1)\times(m-1)}$

so $\begin{matrix} x \\ x \\ x \\ x \end{matrix} \to \begin{matrix} x \\ 0 \\ 0 \\ 0 \end{matrix}$

so $Q_2 Q_1 A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$. Let $Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q}_2 \end{bmatrix}$

why 1st col. not messed up by $Q_2$? since in $\mathbb{R}^{m-1}$, all $\vec{0}$.

finally $Q_n \cdots Q_1 A = R = \begin{bmatrix} x & x & \cdots \\ & x & \vdots \\ 0 & & \end{bmatrix}$. cost? $O(m^2)$ to apply, $n$ times each iter $\approx O(mn^2)$.

triangularization by orthog. steps.

demo: $[Q,R] = qr(A)$. $\to$ $Q$ unitary

how it's done

How get $Q$ st. $A = QR$? $Q = Q_1^{-1} \cdots Q_n^{-1}$
$= Q_1 \cdots Q_n$ why? $(\text{reflection})^2 = I$
$= O(m^2 n)$ or $O(mn)$ to apply $Q^* = Q_n \cdots Q_1$ to a vector $b$.

1) Can prove B.S. : $\tilde{Q}\tilde{R} = A + \delta A$ for $\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_{mach})$

40 mins.

M116 lec 6 pt II · feedback slips ① 4/10/14

FFT:

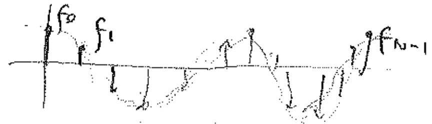
Cooley–Tukey paper 1965 credits: Good '58

but actually ~~...~~ similar in 1850s in UK, for analysing tides, & Runge ~1900.

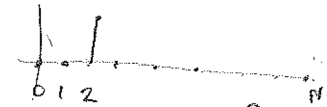
Gauss 1805 treatise, for orbits of asteroids. driven by notice/applications.

Idea of FFT in ~~...~~ notation (not papers):

signal $f \in \mathbb{C}^N$



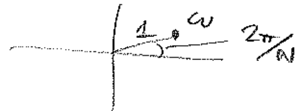
$\xrightarrow{\text{lin. map}}$



$\hat{f} \in \mathbb{C}^N$

← osc twice per signal length

Defn $\omega := e^{\frac{2\pi i}{N}}$



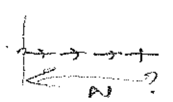
$F^{(N)} \in \mathbb{C}^{N \times N}$ elements $F^{(N)}_{mj} = \omega^{-mj}$

Discrete FT $\hat{f} = F^{(N)} f$ ie $\hat{f}_m = \sum_{j=0}^{N-1} \omega^{-mj} f_j$ for $m = 0, 1 \ldots N-1$ 0-indexed

FFT is the paradigm "fast algorithm": connects to FMM (Zach), RandSVD (Lin). Naive Complexity? $O(N^2)$ matrix-
→ WS?

Sum Lemma $\sum_{k=0}^{N-1} \omega^{jk} = \begin{cases} N & j = 0 \pmod N \\ 0 & \text{otherwise.} \end{cases}$

vs.



HW: prove $\frac{1}{\sqrt{N}} F^{(N)}$ unitary, so thus $\|\hat{f}\| = N \|f\|$ Parseval.

Claim: if eval. DFT formula outside $0 \le m < N$, it's N-periodic: $\hat{f}_{m+kN} = \hat{f}_m \quad \forall m,k \in \mathbb{Z}$

Why? $\hat{f}_{m+kN} = \sum_{j=0}^{N-1} \underbrace{\omega^{-(m+kN)j}}_{\omega^{-mj} \cdot \omega^{-kjN}} f_j$, $\quad \omega^{-kjN} = (\omega^N)^{-kj} = 1$

Say N even: $\hat{f}_m = (F^{(N)} f)_m = \sum_{j=0}^{N-1} \omega^{-mj} f_j = \sum_{k=0}^{N/2-1} \underbrace{\omega^{-m2k}}_{(\omega^2)^{-mk}} f_{2k} + \sum_{k=0}^{N/2-1} \underbrace{\omega^{-m(2k+1)}}_{\omega^{-m}(\omega^2)^{-mk}} f_{2k+1}$

j even ... j odd

shuffle ~~split~~ $f = [f_0 \; f_1 \; f_2 \; f_3 \; \cdots \; f_{N-1}]$ → $f^e$ (even entries), $f^o$ (odd entries)

$f^o, f^e \in \mathbb{C}^{N/2}$

What $\omega$ would appear in length $N/2$ DFT? $e^{\frac{2\pi i}{N/2}} = \omega^2$

②

so, if $0 \le m < N/2$ , $\tilde{f}_m = (F^{(N/2)} f^e)_m + \omega^{-m} (F^{(N/2)} f^o)_m$

1st ~~bottom~~ half of answer

$= \underbrace{\tilde{e}_m + \omega^{-m}}_{\text{twiddle factor}} \tilde{o}_m$ where $\tilde{e} = F^{(N/2)} f^e$, $\tilde{o} = F^{(N/2)} f^o$ half-size DFTs.

2nd half of answer? $N/2 \le m < N$ } use $N/2$-periodicity of $(F^{(N/2)} f^e)_m$ wrt $m$.

$\tilde{f}_{m+N/2} = \tilde{e}_m + \omega^{-(m+N/2)} \tilde{o}_m$

$= \tilde{e}_m - \omega^{-m} \tilde{o}_m$ ← why $\omega^{-N/2} = -1$.

Danielson - Lanczos lemma.

Alg. for length-N DFT of $f$: (N even)

shuffle $f$ into $f^e, f^o$

$\tilde{e} = F^{(N/2)} f^e$

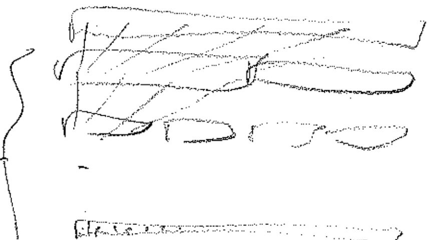
$\tilde{o} = F^{(N/2)} f^o$ } ie do length-$N/2$ DFTs.

stack $\tilde{f} = \begin{bmatrix} \tilde{e} + W\tilde{o} \\ \tilde{e} - W\tilde{o} \end{bmatrix}$  $W = \mathrm{diag}\{\omega^0, \dots \omega^{-N/2+1}\}$

Assume $N = 2^n$ : how apply above? recursion, stop when $f^e, f^o$ length 1 : $\tilde{e} = f^e$, $\tilde{o} = f^o$.

ie $N=2$ DFT is $\tilde{f} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} f$

Complexity?



} N adds & N mults by twiddle facs.

n levels $= \log_2 N$.

total : $O(N \log N)$. , for $N = 10^6$, how much faster? $\sim 10^5 \times$, huge.

Apps: convolving signals (blurring, deblurring), fast multiplication of integers (number theory)

solving PDEs.

extracting components of signals, processing, compression (MP3, JPEG)

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5-10 min feedback Q's.