In class, we considered the function

$$f(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

This function is continuous at all points $(x,y) \neq (0,0)$ because it involves polynomials and square roots, and these are continuous inside their domain.

The function is continuous at $(x,y) = (0,0)$ because we squeeze: observe that

$$\begin{aligned} 0 &\leq y^2 \\ x^2 &\leq x^2 + y^2 \\ |x| &\leq \sqrt{x^2+y^2} \\ \frac{|x|}{\sqrt{x^2+y^2}} &\leq 1 \end{aligned}$$

for all $(x,y) \neq (0,0)$. Therefore

$$|f(x,y)| = \left| \frac{2xy}{\sqrt{x^2+y^2}} \right| = |2y| \left(\frac{|x|}{\sqrt{x^2+y^2}} \right) \leq |2y|$$

so

$$-|2y| \leq f(x,y) \leq |2y|.$$

Since $|2y| \to 0$ as $(x,y) \to (0,0)$, by the squeeze theorem, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

so f is continuous at $(0,0)$.

We will now show that the function f has partial derivatives, but these partial derivatives are not continuous at $(0,0)$. By definition of the partial derivative at $(0,0)$, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{2h\cdot 0 - 0}{\sqrt{h^2+0^2}} \cdot \frac{1}{h} = 0$$

and similarly $f_y(0,0) = 0$. So the function f has its partial derivatives f_x, f_y at $(0,0)$. Read this as saying: if you slice the surface along the x-axis or the y-axis, then you see a horizontal slope at $(0,0)$. In general, this is the method you would use to compute the partial derivatives (using a limit) if you cannot use a formula.

On the other hand, these partial derivatives are *not* continuous at $(0,0)$. For $(x,y) \neq (0,0)$, we can compute the partial derivative using the formula (and the quotient rule):

$$\begin{aligned} f_x(x,y) &= \frac{2y\sqrt{x^2+y^2} - (2xy)\frac{1}{2}(2x)(x^2+y^2)^{-1/2}}{[\sqrt{x^2+y^2}]^2} \\ &= \frac{2y(x^2+y^2) - 2x^2y}{(x^2+y^2)^{3/2}} = \frac{2y^3}{(x^2+y^2)^{3/2}}. \end{aligned}$$

Similarly,

$$f_y(x,y) = \frac{2x^3}{(x^2+y^2)^{3/2}}.$$

But we claim that the limit $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist. Along the path $x = t$ and $y = 0$, we have the limit

$$\lim_{t\to 0} \frac{0}{t^3} = 0.$$

Along the path $x = y = t$, we have the limit

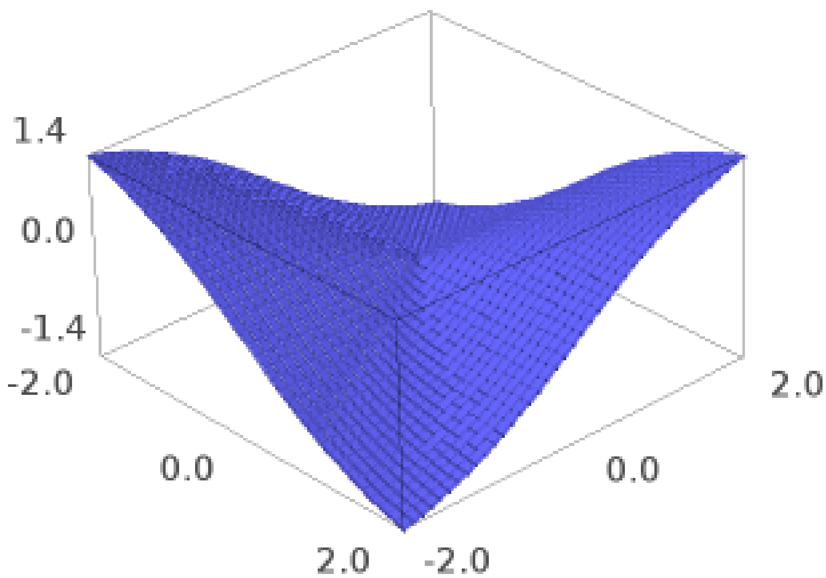
$$\lim_{t\to 0} \frac{2t^2}{(2t^2)^{3/2}} = \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}} \neq 0.$$

Because we obtain different limits along different paths, the limit does not exist.

In particular, this shows that the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ are *not* continuous at $(0,0)$. To check they are continuous, we would need to have

$$f_x(0,0) = \lim_{(x,y)\to(0,0)} f_x(x,y)$$

(and the same with f_y). So although the left-hand side is $f_x(0,0) = 0$ and perfectly fine by the above, the right-hand limit does not exist, so the function cannot be continuous there. (If the limit *had* existed and was equal to 0, then we would conclude that the partial derivative was continuous, but that is not what happened in this case.)



This function is in fact not differentiable at $(0,0)$: it is not nice, it is not smooth, indeed it folds in on itself at this point. More precisely, it is not well-approximated by its "tangent plane" because this tangent plane does not exist: there is no way to have a plane meet the graph at the point $(0,0,0)$ gracing only the point.