Math 11
Fall 2016
Practice Exam I Solutions

1. TRUE or FALSE? (No partial credit; you need not show your work.)

   (a) There exists a vector $\vec{v}$ such that $\langle 2,1,2\rangle \times \vec{v} = \langle 1,-5,2\rangle$.
   FALSE. (Use the dot product to check that $\langle 2,1,2\rangle$ and $\langle 1,-5,2\rangle$ are not orthogonal.)

   (b) For any vectors $\vec{w}$ and $\vec{v}$ we have $\vec{w} \times \text{proj}_{\vec{w}}(\vec{v}) = \vec{0}$ (where $\text{proj}_{\vec{w}}(\vec{v})$ denotes the vector projection).
   TRUE. (The vectors $\vec{w}$ and $\overrightarrow{Proj}_{\vec{w}}(\vec{v})$ are parallel, so their cross product is zero.)

   (c) Any smooth parametrization of the circle $x^2 + y^2 = 1$ gives unit tangent vector at $(1,0)$ equal to $\vec{T} = \langle 0,1\rangle$.
   FALSE. ($\vec{T} = \langle 0,-1\rangle$ is also possible.)

   (d) If $\lim_{x\to 0} f(x,0) = 2$ and $\lim_{y\to 0} f(0,y) = 2$, then $\lim_{(x,y)\to(0,0)} f(x,y) = 2$.
   FALSE. (Checking two lines of approach is not sufficient.)

   (e) It is possible for the intersection of the graph of f with the plane $x = 1$ to have a horizontal tangent line at $(1,2,4)$, and the intersection of the graph of f with the plane $y = 2$ to have a horizontal tangent line at $(1,2,4)$, but for the graph of f not to have a horizontal tangent plane at $(1,2,4)$.
   TRUE. (It may have no tangent plane at all.)

2. Short answer questions. (No partial credit; you need not show your work.)

   (a) Determine whether the lines L_1 with parametric equations

   $$x = 6t \qquad y = 9t \qquad z = -3t + 2$$

   and L_2 with parametric equations

   $$x = -4t - 4 \qquad y = -6t + 3 \qquad z = 2t$$

   are skew, nonparallel but intersecting, parallel, or the same line.

   Solution: They are parallel. The direction vectors $\langle 6,9,-3\rangle$ and $\langle -4,6,2\rangle$ are parallel, since they are scalar multiples of each other. The point $(0,0,2)$ is on L_1, but it is not on L_2. so they are not the same line. (The point on L_2 with $z = 2$ corresponds to $t = 1$, since $z = 2t$, but when $t = 1$ we have $x = -8$, not $x = 0$.)

   (b) Find the arc length of the curve parametrized by $\vec{f}(t) = \langle \cos(t^2), \sin(t^2)\rangle$ for $0 \le t \le \sqrt{2\pi}$.

   Solution: 2π. (The curve is the unit circle.)

(c) A moving object is traveling along the parabola $y = x^2$ in $\mathbb{R}^2$, in the direction of increasing x-coordinate, at a constant speed of 2. When the object is at the point $(0, 0)$, what is its velocity?

Solution: $\vec{v} = \langle 2, 0\rangle$. (Magnitude is speed, direction is tangent to curve.)

3. Find the area of the triangle with vertices $P_0 = (-1-1, 0)$, $P_1 = (0, 1, 1)$, and $P_2 = (0, 0, 3)$.

Solution: This is half the parallelogram with edges $\overrightarrow{P_0P_1}$ and$\overrightarrow{P_0P_2}$, so its area is

$$\frac{1}{2}\left|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}\right| = \frac{1}{2}\left|\langle 1, 2, 1\rangle \times \langle 1, 1, 3\rangle\right| = \frac{1}{2}\left|\langle 5, -2, -1\rangle\right| = \frac{\sqrt{30}}{2}.$$

4. Find all points (x, y) at which the graphs of the functions $f(x, y) = x^2 - 3y^2$ and $g(x, y) = 2x + 2y^3$ have parallel tangent planes.

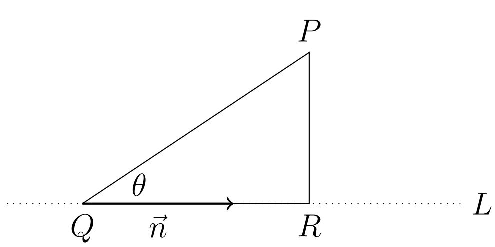
Solution: For the tangent plane to be parallel, the partial derivatives must be the same.

$$f_x(x, y) = 2x \qquad f_y(x, y) = -6y \qquad g_x(x, y) = 2 \qquad g_y(x, y) = 6y^2.$$

Therefore we are looking for points at which $2x = 2$ and $-6y = 6y^2$. The first equation gives $x = 1$ and the second gives $y = 0, y = -1$, so we have two solutions, $(x, y) = (1, 0)$ and $(x, y) = (1, -1)$.

5. Let L be the line containing the point $(1, 1, 1)$ and perpendicular to the plane $3x + 4y - z = 6$. Find the distance from the point $(2, 0, 0)$ to the line L.

Solution: The normal vector to the plane $\vec{n} = \langle 3, 4, -1\rangle$ is parallel to L. We can find the distance from $P = (2, 0, 0)$ to L by letting Q be $(1, 1, 1)$ and R be the point on L closest to $(2, 0, 0)$. The distance we want is the distance between P and R, which we may write as $|PR|$.



In the pictured triangle, the angle at R is a right angle (since the shortest distance from P to L is the perpendicular distance), so $|PR| = |QP| \sin\theta$, where θ is the angle at Q.

$$|PR| = |QP| \sin\theta = \frac{|\vec{n}|\,|QP|\,\sin\theta}{|\vec{n}|} = \frac{|\vec{n} \times \overrightarrow{QP}|}{|\vec{n}|} =$$

$$\frac{|\langle 3,4,-1\rangle \times \langle 1,-1,-1\rangle|}{|\langle -5,2,-7\rangle|} = \frac{|\langle -5,2,-7\rangle|}{|\langle 3,4,-1\rangle|} = \frac{\sqrt{78}}{\sqrt{26}} = \sqrt{3}.$$

6. A spaceship is traveling with position function $\vec{r}(t) = \left\langle \ln(t), \sqrt{2}t, \frac{t^2}{2} \right\rangle$.

   (a) How far does the spaceship travel during the time interval $\frac{1}{2} \le t \le 1$?

   Solution: $\vec{r}\,'(t) = \left\langle \frac{1}{t}, \sqrt{2}, t \right\rangle$, so

$$\frac{ds}{dt} = \sqrt{\frac{1}{t^2} + 2 + t^2} = \sqrt{\left(\frac{1}{t} + t\right)^2} = \frac{1}{t} + t.$$

$$\text{arc length} = \int_{\frac{1}{2}}^{1} \left(\frac{1}{t} + t\right) dt = \left(\ln(t) + \frac{t^2}{2}\right)\Big|_{t=\frac{1}{2}}^{t=1} =$$

$$\left(0 + \frac{1}{2}\right) - \left(-\ln(2) + \frac{1}{8}\right) = \ln(2) + \frac{3}{8}.$$

   (b) At time $t = 1$ the spaceship turns off its engines and continues traveling at constant velocity. Where is it located when $t = 2$?

   Solution: At time $t = 1$ its position is $\vec{r}(1) = \left\langle 0, \sqrt{2}, \frac{1}{2} \right\rangle$ and its velocity is $\vec{r}\,'(1) = \langle 1, \sqrt{2}, 1 \rangle$. Its new position function $\vec{p}(t)$ satisfies $\vec{p}\,'(t) = \langle 1, \sqrt{2}, 1 \rangle$ and $\vec{p}(1) = \left\langle 0, \sqrt{2}, \frac{1}{2} \right\rangle$. This function is

$$\vec{p}(t) = \left\langle 0, \sqrt{2}, \frac{1}{2} \right\rangle + (t-1)\left\langle 1, \sqrt{2}, 1 \right\rangle.$$

   We can find this by knowing that we should use the tangent approximation to $\vec{r}(t)$ near $t = 1$, or by integrating $\vec{p}\,'(t) = \langle 1, \sqrt{2}, 1 \rangle$ and using the initial condition $\vec{p}(1) = \left\langle 0, \sqrt{2}, \frac{1}{2} \right\rangle$ to solve for the constant of integration.

7. Suppose that C is a level curve of the differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, and $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ is a smooth parametrization of C.

   (a) Explain how we know that

$$\frac{d}{dt}(f(\vec{r}(t))) = 0.$$

   Solution: Since the value of f is constant on C, which is the range of $\vec{r}$, the value of $f(\vec{r}(t))$ is constant, so its derivative is zero.

(b) If $(a, b) = \vec{r}(d)$ is a point on C, what does $\vec{r}\,'(d)$ tell us about C?

Solution: It tells us the direction of a tangent vector to C at the point (a, b).

(c) Use parts (a) and (b) and the Chain Rule to prove that if $(a, b) = \vec{r}(d)$ is a point on C, and if $\left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle$ is nonzero, then $\left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle$ must be perpendicular to C.

Solution: To show that $\left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle$ is perpendicular to C, we will show that it is perpendicular to the vector $\vec{r}\,'(d)$, which is tangent to C by part (b). To show this, we will show that their dot product is zero. Using the Chain Rule:

$$\left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle \cdot \vec{r}\,'(d) = \left\langle \frac{\partial f}{\partial x}(\vec{r}(d)), \frac{\partial f}{\partial y}(\vec{r}(d)) \right\rangle \cdot \frac{d\vec{r}}{dt}(d) = \frac{d}{dt}(f(\vec{r}(d))).$$

By part (a), this is zero, which is what we needed to show.

If you prefer to use the Chain Rule in a different form:
Write $w = f(x, y)$ and $\langle x, y \rangle = \vec{r}(t)$. Then we have (evaluating everything at $x = a, y = b, t = d$)

$$\left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle \cdot \vec{r}\,'(d) = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle =$$

$$\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \frac{dw}{dt} = \frac{d}{dt}(f(\vec{r}(d))).$$

8. Each function matches exactly one of the pictures on the next page, either a graph or a contour plot. Identify the picture that goes with each function.

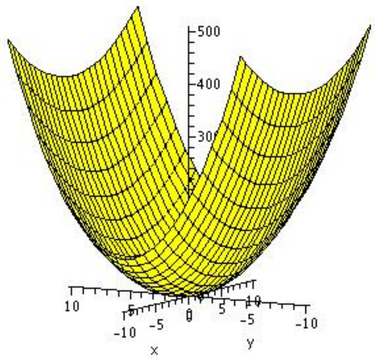
(a) $f(x, y) = x^2 + 4y^2$
(b) $f(x, y) = x^2 + 2xy + y^2$
(c) $f(x, y) = 4x^2 - 2y^2$

Solution:

(a) $\boxed{A.}$ The intersection with the xz-plane is a parabola, and level curves are ellipses, ruling out B, C, E, F. Sketching the level curve $x^2 + 4y^2 = 1$ rules out D, since the ellipses are the wrong shape.

(b) $\boxed{E.}$ This is $(x + y)^2$, so the level curves are lines $x + y = c$, ruling out A, B, C, D. Sketching level curves for $f(x, y) = 0, 1, 2$ shows that the spacing rules out F.

(c) $\boxed{B.}$ The level curves are hyperbolae. (Alternatively, the intersections with the coordinate planes are an upward-facing parabola, a downward-facing parabola, and the crossed lines $y = \pm\sqrt{2}\,x$.)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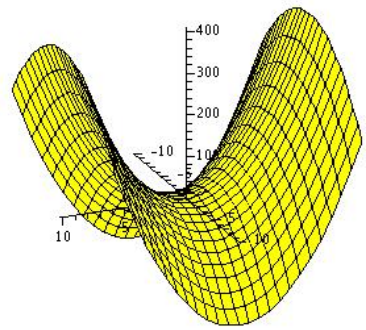
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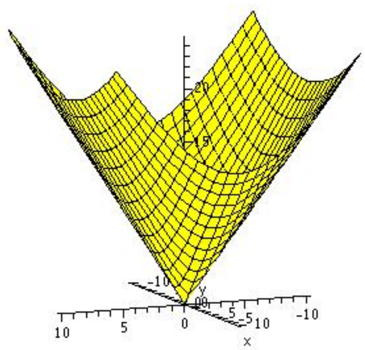
A **B**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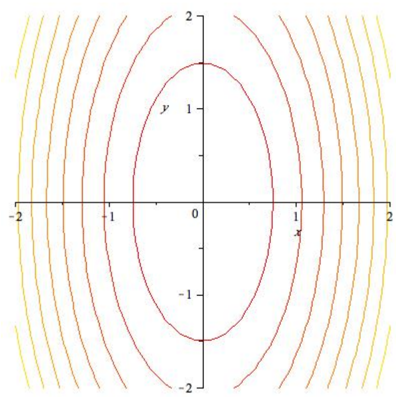
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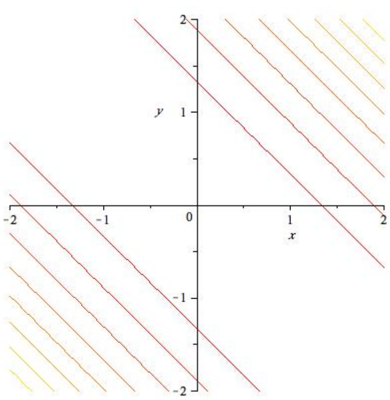
C **D**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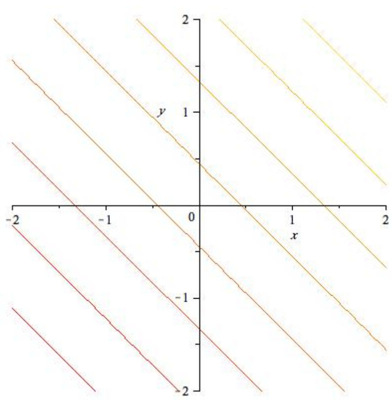
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E **F**