Math 11
Fall 2016
Exam II Practice Sample Solutions

1. TRUE or FALSE? (Graded on answer only; you need not show your work.)

   (a) If (a, b) is a critical point of $f(x, y)$, the second partial derivatives of f are continuous, and

   $$f_{xx}(a, b) = 2 \quad f_{xy}(a, b) = 2 \quad f_{yx}(a, b) = 2 \quad f_{yy}(a, b) = 2,$$

   then (a, b) cannot be a local maximum point of f.

   Solution: FALSE. Knowing the discriminant is zero tells us nothing about whether the function has a local maximum point.

   (b) For any differentiable functions f and g from $\mathbb{R}^2$ to $\mathbb{R}$, we have $\nabla(f+g) = \nabla f + \nabla g$.

   Solution: TRUE.

   (c) If f is a continuous function with continuous partial derivatives defined on the unit disc D given by $x^2 + y^2 \leq 1$, and $\nabla f(1, 0) = \langle 1, 1 \rangle$, then it is possible that f attains its maximum value on D at the point $(1, 0)$.

   Solution: FALSE. The gradient of the constraint function for the boundary of the disc at $(1, 0)$ is $\langle 2, 0 \rangle$ and this vector is not parallel to $\langle 1, 1 \rangle$, the gradient of f at $(1, 0)$.

   (d) If f is a continuous function with continuous partial derivatives and $\nabla f(0, 0) = \langle 1, 0 \rangle$, then for any unit vector $\vec{u}$ we have

   $$(D_{\vec{u}} f)(0, 0) \leq \frac{\partial f}{\partial x}(0, 0).$$

   Solution: TRUE. The direction of the gradient tells us that the maximum directional derivative is in the direction of **i**; that is, the maximum directional derivative is the partial derivative with respect to x.

   (e)
   $$\int_0^1 \int_0^y x^2 \, dx \, dy = \int_0^y \int_0^1 x^2 \, dy \, dx.$$

   Solution: FALSE. The second integral makes no sense; the outer limits on x must be constants.

2. Short answer questions. Parts (a) and (b) have nothing to do with each other. (Graded on answer only; you need not show your work.)

(a) Rewrite

$$\iint_D (x+y)\,dA,$$

where D is the parallelogram with vertices $(0,0)$, $(2,1)$, $(3,4)$, and $(1,3)$, as an integral in the form

$$\int_a^b \int_c^d f(u,v)\,du\,dv$$

by using a suitable change of variables.

Solution: Use a linear transformation T with $T(0,0) = (0,0)$, $T(1,0) = (2,1)$, $T(0,1) = (1,3)$; namely $T(u,v) = (2u+v, u+3v)$. Then the Jacobian is

$$\det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 5$$

and the integral is

$$\int_0^1 \int_0^1 ((2u+v)+(u+3v))5\,dv\,du = \int_0^1 \int_0^1 (3u+4v)5\,dv\,du$$

Note: This is not the only possible solution. Any solution leading to an integral in which the limits on u or v are constants would be fine.

(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with continuous partial derivatives, and $\nabla f(1,2) = \langle 3,4 \rangle$.

i. What is the directional derivative of f at $(1,2)$ in the direction given by $\langle -4,3 \rangle$?
Solution: 0, since this direction is orthogonal to the gradient.

ii. What is the minimum possible value of a directional derivative $D_{\vec{u}} f(1,2)$?
Solution: $-|\nabla f(1,2)| = -5$.

3. Find the absolute maximum and absolute minimum values of

$$h(x,y,z) = x^2 + y^2 - 4x + 6y + 2z^2 - 6$$

on the region

$$R = \{(x,y) : x^2 + y^2 + z^2 = 4\}.$$

Solution: R is a level surface of $g(x,y,z) = x^2+y^2+z^2$, so use Lagrange multipliers:

$$\langle 2x-4, 2y+6, 4z \rangle = \lambda \langle 2x, 2y, 2z \rangle \qquad x^2+y^2+z^2 = 4$$

has solutions

$$x = \frac{4}{\sqrt{13}} \qquad y = -\frac{6}{\sqrt{13}} \qquad z = 0 \qquad \lambda = 1 - \frac{\sqrt{13}}{2}$$

and

$$x = -\frac{4}{\sqrt{13}} \qquad y = \frac{6}{\sqrt{13}} \qquad z = 0 \qquad \lambda = 1 + \frac{\sqrt{13}}{2}.$$

The minimum and maximum values of h on R are therefore

$$h\left(\frac{4}{\sqrt{13}}, -\frac{6}{\sqrt{13}}, 0\right) = -4\sqrt{13} - 2 \approx -16.4222,$$

$$h\left(-\frac{4}{\sqrt{13}}, \frac{6}{\sqrt{13}}, 0\right) = 4\sqrt{13} - 2 \approx 12.4222.$$

4. Find all critical points of the function $f(x, y) = x^3 - 3xy + y^3$, and classify them as local maxima, local minima, or saddle points.

   Solution:

   The first partial derivatives exist everywhere, and are given by $f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$. We have $f_x = f_y = 0$ at the points $(0, 0)$ and $(1, 1)$, so the critical points of f are $(0, 0)$ and $(1, 1)$.

   At $(0, 0)$ we have $f_{xx}f_{yy} - f_{xy}f_{yx} = -9 < 0$, so $(0, 0)$ is a saddle point.

   At $(1, 1)$ we have $f_{xx} = 6 > 0$ and $f_{xx}f_{yy} - f_{xy}f_{yx} = 27 > 0$, so $(1, 1)$ is a local minimum.

5. Find

$$\iint_D xy \, dA$$

   where D is the region in the first quadrant between the curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

   Solution: Use polar coordinates.

$$\int_0^{\frac{\pi}{2}} \int_1^2 (r\cos\theta)(r\sin\theta)r \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \frac{15}{4}(\cos\theta)(\sin\theta) \, dr \, d\theta = \frac{15}{8}\sin^2\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{15}{8}.$$

6. Two surfaces S and T are given in spherical coordinates by

$$\text{surface } S: \quad \phi = \frac{\pi}{3}$$

$$\text{surface } T: \quad \rho = 4\cos\phi.$$

   (a) Describe the surfaces S and T.

   Solution: S is an upward-facing cone with vertex at the origin, and T is a sphere of radius 2 with center $(0, 0, 1)$.

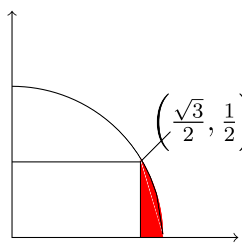
(b) Find the volume of the solid that lies above S and below T.

Solution:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 10\pi.$$

7. Sketch the region of integration, and rewrite the integral, first with the opposite order of integration, and then as an integral in polar coordinates.

$$\int_0^{\frac{1}{2}}\int_{\frac{\sqrt{3}}{2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{x^2+y^2}} \, dx \, dy.$$



$$\int_{\frac{\sqrt{3}}{2}}^{1}\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}} \, dy \, dx = \int_0^{\frac{\pi}{6}}\int_{\frac{\sqrt{3}}{2\cos\theta}}^{1} dr \, d\theta.$$

8. Evaluate the triple integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2} \sqrt{x^2+y^2} \, dz \, dy \, dx.$$

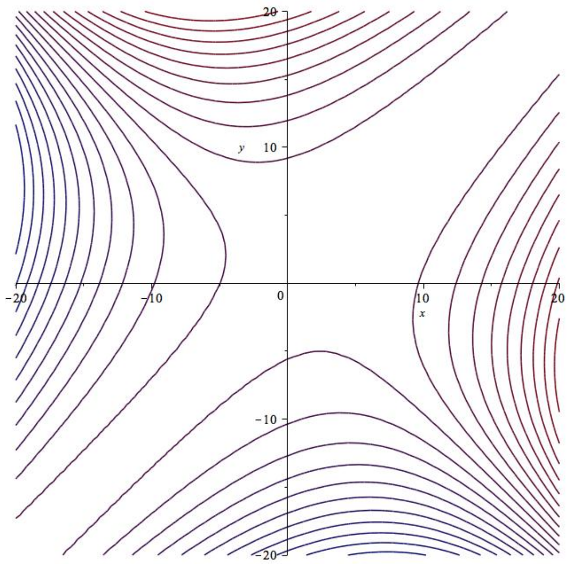
Solution: Convert to cylindrical coordinates to get the bounds

$$0 \le \theta \le 2\pi, 0 \le r \le 1, r^2 \le z \le 2 - r^2.$$

Then the integral becomes

$$\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r^2 \, dz \, dr \, d\theta = 2\int_0^{2\pi} (r^3/3 - r^5/5)\Big|_{r=0}^{r=1} d\theta = \frac{8\pi}{15}.$$

9. Short answer question. (Graded on answer only; you need not show your work.)



The picture shows the contour plot of a function $f(x, y)$. In the region $x > 5$, $y < -5$, both f_x and f_y are positive.

Does $f_{xx}(10, -15)$ appear to be positive or negative?

Solution: Positive. Since $f_x > 0$ in that region, as we move to the right (in the direction of increasing x) we are moving uphill. We cross contour lines at closer and closer intervals, therefore the slope in the x direction f_x is increasing. Since we are moving in the x-direction, this rate of increase of slope is f_{xx}.

Does $f_{yy}(10, -15)$ appear to be positive or negative?

Solution: Negative, by similar reasoning. We are moving uphill but the slope of our path is decreasing.