Three-dimensional coordinate system (x-axis points out of paper):
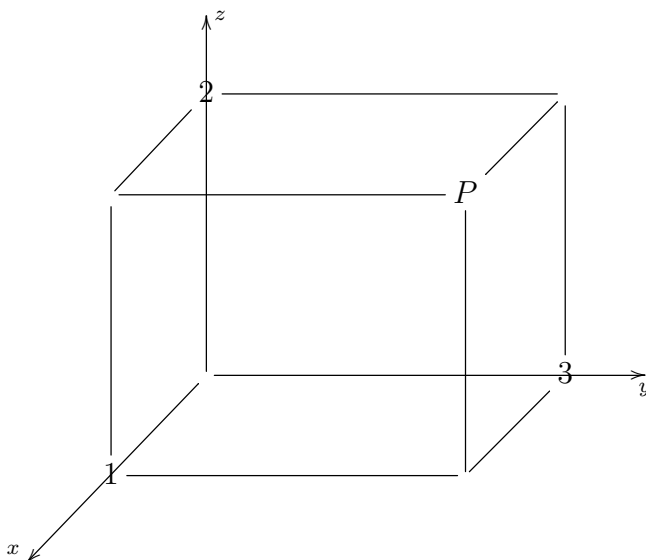
$$P = (1, 3, 2)$$

Distance between points:
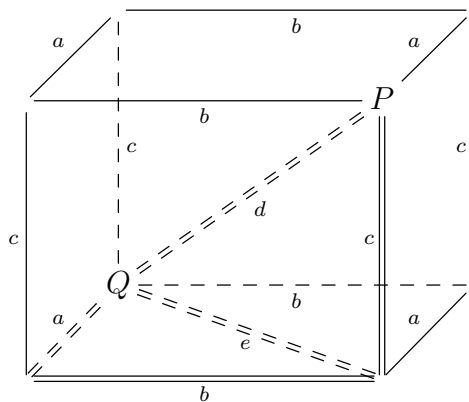
$$d^2 = e^2 + c^2 = (a^2 + b^2) + c^2 \qquad d = \sqrt{a^2 + b^2 + c^2}$$

Distance from $Q = (x_1, y_1, z_1)$ to $P = (x_2, y_2, z_2)$ is

$$\boxed{|QP| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$\mathbb{N} = \{0, 1, 2 \dots\}$ is the set of natural numbers (N is for number).

$\mathbb{Z} = \{\dots -2, -1, 0, 1, 2 \dots\}$ is the set of all integers (Z is for *Zahlen*, number in German).

$\mathbb{Q}$ is the set of rational numbers, or fractions (Q is for quotient).

$\mathbb{R}$ is the set of all real numbers (yes, R is for real).

$\mathbb{R}^3$ denotes 3-dimensional space, or the set of all triples (a, b, c) of real numbers.

Equations and inequalities in $\mathbb{R}^3$:

$z = 0$ Horizontal plane (xy-plane)

$z = 3$ Horizontal plane

$x = 1$ Vertical plane (parallel to screen, or yz-plane)

$x = y$ Vertical plane (diagonal to yz-plane)

$0 \leq y < 1$ Vertical slab (between plane $y = 0$ and plane $y = 1$, including first but not second)

$x^2 + y^2 = 4$ Cylinder (radius 2, around z-axis)

$4 \leq x^2 + y^2 \leq 9$ Cylindrical shell (inner radius 2, outer radius 3, around z-axis, including surfaces)

$y = x^2$ Vertical curved surface with parabolic cross-section

$x^2 + y^2 + z^2 = 4$ Sphere (radius 2, around origin)

$x^2 + y^2 + z^2 \leq 4$ Solid ball (radius 2 around origin)

$(x - 1)^2 + (y - 2)^2 + (z + 1)^2 = 1$ Sphere (radius 1, around $(1, 2, -1)$)

Sphere of radius r with center (a, b, c):

$$|(x, y, z)(a, b, c)| = r$$

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r$$

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2}$$

Example: Suppose S is a sphere, the center of S is $(1, -1, 2)$, and the point $(2, 3, 0)$ lies on S. Find an equation for S.

Radius of S is distance from center to sphere, so distance from $(1, -1, 2)$ to $(2, 3, 0)$.

$$r = \sqrt{(2-1)^2 + (3-(-1))^2 + (0-2)^2} = \sqrt{1^2 + 4^2 + 2^2} = \sqrt{21}$$

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

$$(x-1)^2 + (y+1)^2 + (z-2)^2 = 21$$

Example: Find the center and radius of the sphere with equation

$$x^2 + 2x + y^2 + z^2 - 6z = 6$$

$$x^2 + 2x + 1 - 1 + y^2 + z^2 - 6z + 9 - 9 = 6$$

$$(x^2 + 2x + 1) - 1 + y^2 + (z^2 - 6z + 9) - 9 = 6$$

$$x^2 + 2x + 1 + y^2 + z^2 - 6z + 9 = 16$$

$$(x+1)^2 + y^2 + (z-3)^2 = 16$$

$$\text{center} = (-1, 0, 3) \quad \text{radius} = 4$$

Problem: Find equations for two different spheres that intersect the plane $z = 3$ in the circle $x^2 + y^2 = 16$.
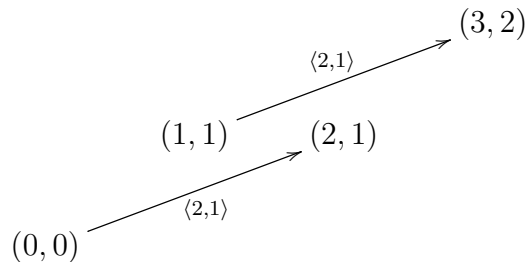
Vectors:
We will think about vectors in 3 ways: algebra, geometry, and applications.

Algebra: A vector in $\mathbb{R}^n$ is an n-tuple of real numbers $\langle a_1, a_2, \ldots, a_n \rangle$. (We often use angle brackets to distinguish vectors from points.)

For us, usually $n = 2$ or $n = 3$.

Geometry: A vector $\langle a_1, a_2, \ldots, a_n \rangle$ can be drawn as an arrow from the point $(0, 0, \ldots, 0)$ to the point $(a_1, a_2, \ldots, a_n)$ OR it can be drawn as an arrow of the same length and direction starting at any other point.

Two arrows with the same length and direction are two pictures of the same vector.

Going along an arrow representing $\langle a_1, a_2, a_3 \rangle$ you go a_1 units in the x-direction, a_2 units in the y-direction, and a_3 units in the z-direction.

Applications: Vectors are used to model anything that is given by its magnitude (represented by the length of the vector) and direction (represented by the direction of the vector).

Examples:

1. The position vector of the point P with coordinates (a, b, c) is the vector $\langle a, b, c \rangle$ from the origin to P.

   Note: The position vector of P is a vector, so it can be drawn starting at any point. When it is drawn starting at the origin, it ends at P.

2. If an object moves from point P to point Q, its *displacement* is represented by the vector that can be drawn with tail at P and head at Q.

   Displacement gives direction of motion and distance, but not starting position. So an object that moves from $(0, 0, 0)$ to $(1, 1, 1)$ and an object that moves from $(1, 2, 3)$ to $(2, 3, 4)$ have the same displacement, $\langle 1, 1, 1 \rangle$.

3. Velocity is represented by a vector. Its direction is the direction of motion, and its length gives the speed.

4. Force is represented by a vector.

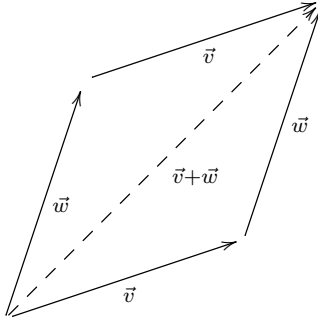5. Acceleration is represented by a vector.

   Note: An object can have constant speed but nonzero acceleration, if its direction of motion is changing. We will be able to analyze acceleration later.

Vector addition: (Add two vectors, result is a vector.)

Algebra: Add vectors coordinatewise.

$$\langle a_1, b_1, c_1 \rangle + \langle a_2, b_2, c_2 \rangle = \langle a_1 + a_2, b_1 + b_2, c_1 + c_2 \rangle$$

Geometry: Parallelogram law.

Applications:
Net displacement, resultant force, net velocity.

Scalar multiplication: (Multiply a vector by a real number, result is a vector.)

Algebra: Multiply coordinates by the real number (scalar).

$$k \langle a_1, b_1, c_1 \rangle = \langle ka_1, kb_1, kc_1 \rangle$$

Geometry: The length of $k\vec{v}$ is $|k|$ times the length of $\vec{v}$. The direction of $k\vec{v}$ is the same as the direction of $\vec{v}$ if $k > 0$, and the opposite if $k < 0$.
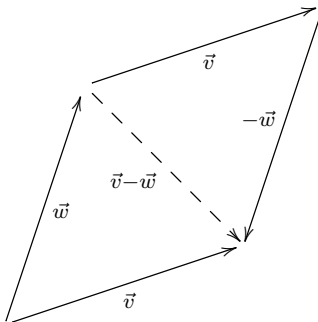
Subtraction: (Subtract one vector from another, result is a vector.)

Algebra:
$$-\vec{v} = (-1)\vec{v} \quad \text{and} \quad \vec{v} - \vec{w} = \vec{v} + (-\vec{w})$$
$$\langle a_1, b_1, c_1 \rangle - \langle a_2, b_2, c_2 \rangle = \langle a_1 - a_2, b_1 - b_2, c_1 - c_2 \rangle$$

Geometry: A different parallelogram law.

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Magnitude (norm) of vector: (A real number, or scalar.)

Geometry: $|\vec{v}|$ is the length of $\vec{v}$.
Algebra: Use the distance formula:

$$|\langle a_1, b_1, c_1 \rangle| = \sqrt{(a_1)^2 + (b_1)^2 + (c_1)^2}$$

Applications: If $\vec{v}$ represents displacement, $|\vec{v}|$ represents distance; if $\vec{v}$ represents velocity, $|\vec{v}|$ represents speed; if $\vec{v}$ represents force, $|\vec{v}|$ represents the magnitude of the force.

Zero vector:

The zero vector is the vector $\vec{0} = \langle 0, 0, 0 \rangle$ whose coordinates are all 0.
The zero vector has length 0 and has no direction.

Unit vector:

A unit vector is a vector whose magnitude is 1.
We often use unit vectors to specify direction.

Parallel vectors:

Geometry: Two nonzero vectors are parallel if they have either the same direction or opposite directions.
Algebra: Two nonzero vectors are parallel if one is a scalar multiple of the other.

Notation: Sometimes vectors are written with an arrow on top, $\vec{v}$. Sometimes, instead, they are written in boldface, **v**.
Sometimes the norm of a vector $\vec{v}$ is written $\|\vec{v}\|$ instead of $|\vec{v}|$.
Some texts that always use boldface for vectors, use v for the norm of **v**.

Example: Find a unit vector whose direction is the direction of motion of an object moving in a straight line from $(0, 1, 0)$ to $(3, 5, 12)$.

Displacement vector: $\vec{v} = \langle 3, 4, 12 \rangle$
Vector in the same direction: $\vec{u} = k\vec{v} = \langle 3k, 4k, 12k \rangle$ where $k > 0$.
Magnitude of $\vec{u}$ is $|\vec{u}| = k|\vec{v}|$. We want $|\vec{u}| = 1$:

$$|\vec{u}| = k|\vec{v}|$$

$$1 = k|\vec{v}| = k|\langle 3, 4, 12 \rangle| = k\sqrt{3^2 + 4^2 + 12^2} = k\sqrt{169} = 13k$$

$$k = \frac{1}{|\vec{v}|} = \frac{1}{13}$$

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$$

Theorem: If $\vec{v}$ is a nonzero vector, the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{1}{|\vec{v}|}\,\vec{v}.$$

Standard basis for $\mathbb{R}^2$:

$$\{\hat{i}, \hat{j}\}$$
$$\hat{i} = \langle 1, 0 \rangle$$
$$\hat{j} = \langle 0, 1 \rangle$$
$$\langle a, b \rangle = a\hat{i} + b\hat{j}$$
$$\vec{e_1} = \hat{i} \quad \vec{e_2} = \hat{j}$$

Standard basis for $\mathbb{R}^3$:

$$\{\hat{i}, \hat{j}, , \hat{k}\}$$
$$\hat{i} = \langle 1, 0, 0 \rangle$$
$$\hat{j} = \langle 0, 1, 0 \rangle$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$
$$\langle a, b, c \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$
$$\vec{e_1} = \hat{i} \quad \vec{e_2} = \hat{j} \quad \vec{e_3} = \hat{k}$$

Standard basis for $\mathbb{R}^n$:

$$\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}\}$$
$$\vec{e_i} = \left\langle 0, \ldots, 0, \underbrace{1}_{i^{th}\ \text{position}}, 0 \ldots, 0 \right\rangle$$

Notation: It is common, particularly in physics and engineering, to write the vector $\langle a, b, c \rangle$ as $a\hat{i} + b\hat{j} + c\hat{k}$, or as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Example: We set up our axes with the z-axis vertical, the x-axis pointing east, and the y-axis pointing north. An airplane's velocity relative to the air around it is $\langle 150, 200, 5 \rangle$, with units of miles and hours. If the wind is blowing from the northeast at 15 miles per hour (and parallel to the ground), what is the airplane's velocity relative to the ground?

Example: This example uses SI units of meters and seconds. Find a vector that represents the velocity of an object moving in a straight line from the origin toward the point $(15, 20, 60)$ (where coordinates are in meters) at a speed of .5 meters per second. (Suggestion: First find a unit vector in the direction of motion. Then find the velocity vector.)

If the object starts from the origin at time $t = 0$, where will it be at time $t = 6$? (Suggestion: Find its displacement, by finding its direction of motion and the distance it travels.)

Problem: If an object starts at a point with position vector $\vec{r}_0$ and moves with velocity $\vec{v}$ for t units of time, what is the position vector of its final location?

Cultural enrichment:

Definition: A *vector space* (over $\mathbb{R}$) consists of objects that can be added to each other and multiplied by real numbers (scalars). There is an object called $\mathbf{0}$, and for each object $\mathbf{x}$ there is an associated object $-\mathbf{x}$. Addition and scalar multiplication follow these rules:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

$$\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$$

$$\mathbf{x} + \mathbf{0} = \mathbf{x}$$

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$$

$$1\mathbf{x} = \mathbf{x}$$

$$a(b\mathbf{x}) = (ab)\mathbf{x}$$

$$a(\mathbf{x} + \mathbf{y}) = (a\mathbf{x}) + (\mathbf{y})$$

$$(a + b)\mathbf{x} = (a\mathbf{x}) + (b\mathbf{x})$$

Examples of vector spaces:
$\mathbb{R}^3$
$\mathbb{R}^n$
The set of all 2×3 matrices with real number entries.
The set of all quadratic polynomials with real number coefficients.
The set of all polynomials with real number coefficients.
The set of all continuous functions from $[0, 1]$ to $[0, 1]$.
The complex numbers $\mathbb{C}$.

$\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$ spans a vector space if every $\mathbf{x}$ in the vector space can be expressed as

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n$$

where $a_1, a_2, \ldots, a_n$ are real numbers.

$\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$ is a *basis* for a vector space if it spans the vector space, but no smaller subset does.

A vector space has dimension n if it has a basis of size n.

Examples:
$\mathbb{R}^3$ has dimension 3 (one basis is $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$); $\mathbb{R}^n$ has dimension n.
The set of quadratic polynomials has dimension 3. A basis is $\{1, x, x^2\}$.
The set of 2×3 matrices has dimension 6. The matrices with a 1 in one entry and 0 in all other entries comprise a basis.
$\mathbb{C}$ has dimension 2. A basis is $\{1, i\}$.
The other examples above have infinite dimension.