Math 11
Fall 2016
Section 1
Monday, October 3, 2016

First, some important points from the last class:

Definition: If $f : \mathbb{R}^n \to \mathbb{R}$, the *gradient* of f is the vector whose components are its partial derivatives:

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}(x,y,z),\ \frac{\partial f}{\partial y}(x,y,z),\ \frac{\partial f}{\partial z}(x,y,z) \right\rangle.$$

If f is differentiable, we may also calll ∇f the total derivative of f.

Theorem (the chain rule): If $\vec{r}(t)$ is differentiable at t_0, and $f(x,y,z)$ is differentiable at $\vec{r}(t_0)$, then

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z \approx \frac{\partial w}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial w}{\partial y}\frac{dy}{dt}\Delta t + \frac{\partial w}{\partial z}\frac{dz}{dt}\Delta t$$

The chain rule in different settings:

$$t \to x \to w$$

$$\frac{dw}{dt} = \frac{dw}{dx} + \frac{dx}{dt}$$

$$t \to (x,y,z) \to w$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$(s,t) \to (x,y,z) \to w$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

Warm-up Problems: Let $f(x, y) = ax + by + d$, and let S be the graph of f. Note that S is a plane.

Two points P and Q lie on S. The coordinates of P are (x, y, z) and the coordinates of Q are $(x + \Delta x, y + \Delta y, z + \Delta z)$. Find Δz as a function of Δx and Δy.

$$\Delta z = \boxed{a\Delta x + b\Delta y}$$

Express $\overrightarrow{PQ}$ as the sum of two vectors, $\vec{w}_H$ horizontal (with z-component equal to 0) and $\vec{w}_V$ vertical (with x- and y-components equal to 0).

$$\vec{w}_H = \boxed{\langle \Delta x, \Delta y, 0\rangle} \qquad \vec{w}_V = \langle 0, 0, \Delta z\rangle = \boxed{\langle 0, 0, a\Delta x + b\Delta y\rangle}$$

If $\langle \Delta x, \Delta y\rangle = h\,\langle \cos\theta, \sin\theta\rangle$, find the "slope" (vertical rise over horizontal run) of $\overrightarrow{PQ}$.

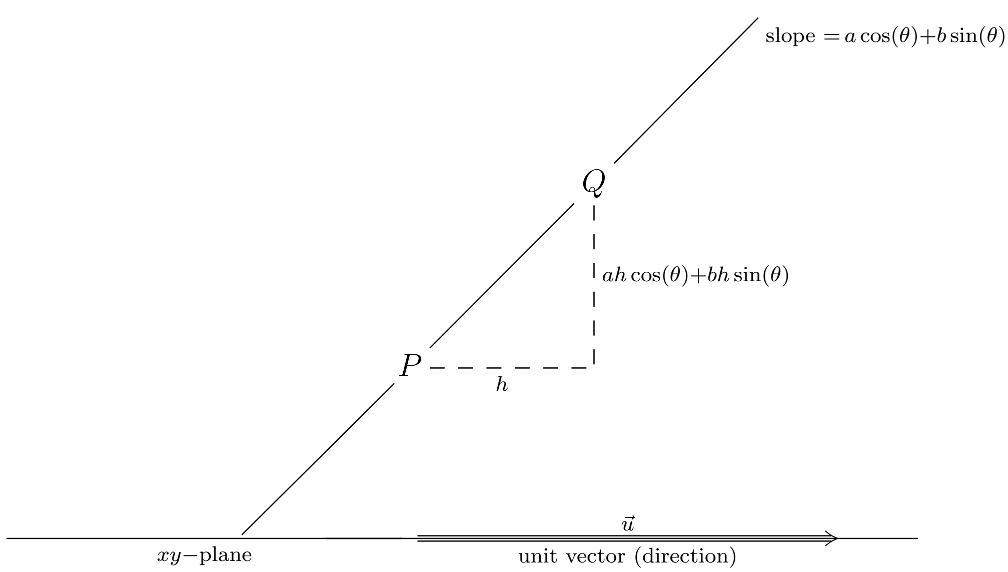
$$\text{slope} = \frac{\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{ah\cos(\theta) + bh\sin(\theta)}{\sqrt{h^2\cos^2(\theta) + h^2\sin(\theta)}} = \boxed{a\cos(\theta) + b\sin(\theta)}$$

This is

$$\boxed{\langle a, b\rangle \cdot \langle \cos(\theta), \sin(\theta)\rangle\,.}$$

Note $a = f_x$ and $b = f_y$ so $\langle a, b\rangle = \nabla f$.

Explain why this is a kind of partial derivative of f, in the direction given by the unit vector $\langle \cos(\theta), \sin(\theta)\rangle$.



Definition: If $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{u}$ is a unit vector in $\mathbb{R}^n$, then the *directional derivative* of f at $\vec{x}$ in the direction $\vec{u}$ is

$$\underbrace{D_{\vec{u}} f(\vec{x}) = \frac{\partial f}{\partial \vec{u}}(\vec{x})}_{\text{notation}} = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}.$$

This is the rate of change of f with respect to distance in the domain, when the argument (input) of f is moving in the direction $\vec{u}$.

If $f : \mathbb{R}^2 \to \mathbb{R}$, then $D_{\vec{u}}(x, y)$ is the slope of the slice of the graph of f in the vertical plane containing the line in the xy-plane through the point (x, y) in the direction of the vector $\vec{u}$.

Theorem: If f is differentiable at $\vec{x}$, then

$$D_{\vec{u}}(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}.$$

Warning: The vector $\vec{u}$ must be a unit vector.

Example: For differentiable $f : \mathbb{R}^2 \to \mathbb{R}$,

$$D_{\hat{\imath}} f(x, y) = \nabla f(x, y) \cdot \hat{\imath} = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle \cdot \langle 1, 0 \rangle = \frac{\partial f}{\partial x}(x, y).$$

Informal proof: We showed in the warm-up problems that this theorem holds for a linear function L, whose graph is a plane.

If f is differentiable at $\vec{x}$, then there is a linear function L, with the same partial derivatives, whose graph is tangent to the graph of f at $\langle \vec{x}, f(\vec{x}) \rangle$. But if the graphs are tangent, they have the same slope in every direction. That is, they have the same directional derivatives.

This informal proof can be made completely formal, by using the limit definition of directional derivative and the limit definition of tangent. There is a different proof in the exercises.

Example: Suppose $\nabla f(x,y) = \langle 3, 4 \rangle$. What is the largest possible value of $D_{\vec{u}} f(x,y)$, and for what value of $\vec{u}$ do we get this value?

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} = |\nabla f(x,y)|\,|\vec{u}| \cos\theta = |\nabla f(x,y)| \cos\theta,$$

where θ is the angle between $\nabla f(x,y)$ and $\vec{u}$.

The maximum possible value of the directional derivative is $|\nabla f(x,y)|$, which we get when $\cos(\theta) = 1$, or $\theta = 0$, or $\vec{u}$ is in the same direction as $\nabla f(x,y)$.

In our case, the maximum possible value is $|\langle 3, 4 \rangle| = 5$, which occurs when $\vec{u}$ is in the direction of $\langle 3, 4 \rangle$, or $\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

Example: In the same situation, for what values of $\vec{u}$ is $D_{\vec{u}} f(x,y) = 0$?

When $\cos(\theta) = 0$, or $\vec{u} \perp \nabla f(x,y)$.
In our case, this happens at $\vec{u} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$ and $\vec{u} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$.

Theorem: If f is differentiable at $(\vec{x})$ then:

The maximum value of $D_{\vec{u}}f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ points in the direction of $\nabla f(\vec{x})$.

The minimum value of $D_{\vec{u}}f(\vec{x})$ is $-|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ points in the opposite direction to $\nabla f(\vec{x})$.

The value of $D_{\vec{u}}f(\vec{x})$ is 0 when $\vec{u}$ is perpendicular to $\nabla f(\vec{x})$.

The vector $\nabla f(\vec{x})$ is normal to the level set (level curve or level surface) of f containing the point $(\vec{x})$. (Intuitively, if you are moving on a smooth surface, the straight uphill direction should be at right angles to the horizontal direction.)

Example: Our famous crawling bug is on the graph of the function $f(x, y) = x^2y - xy^2 + 30$, and its shadow in the xy-plane is at the point $(1, 2)$. If the bug crawls uphill as steeply as possible, in what direction will its shadow be moving?

$$\nabla f(x, y) = \left\langle 2xy - y^2,\, x^2 - 2y \right\rangle \qquad \nabla f(1, 2) = \langle 0,\, -3\rangle$$

Its shadow will move in the direction given by $\nabla f(1, 2) = \langle 0, -3\rangle$, that is, in the direction of the unit vector $\langle 0, -1\rangle$.

In what direction will the bug itself be moving?

The slope of the bug's path will be $|\nabla f(1, 2)| = |\,\langle 0, -3\rangle\,| = 3$.

A vector in the direction of the bug's motion is $\langle 0, -1, 3\rangle$ (moving a distance 1 in the direction of $\nabla f(1, 2)$ in the xy-plane, and a vertical distance of $|\nabla f(1, 2)| = 3$).

A unit vector giving the direction of the bug's motion is $\left\langle 0,\, \dfrac{1}{\sqrt{10}},\, -\dfrac{3}{\sqrt{10}} \right\rangle$

Proof that $\nabla f(\vec{x}_0)$is normal to the level set S of f at $\vec{x}_0$: Let γ be any smooth curve lying in S and passing through $\vec{x}_0$. Let $\vec{r}$ be a smooth parametrization of γ with $\vec{r}(t_0) = \vec{x}_0$.

Since γ lies in S, we have that f is constant on γ, and so $f(\vec{r}(t))$ is constant. Applying the Chain Rule, we see

$$0 = (f \circ \vec{r})'(t_0) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0) = \nabla f(\vec{x}_0) \cdot \vec{r}\,'(t_0).$$

That is $\nabla f(\vec{x}_0)$ and $\vec{r}\,'(t_0)$ are orthogonal.

Since $\vec{r}\,'(t_0)$ is tangent to γ at $\vec{x}_0$, this shows $\nabla f(\vec{x}_0)$ is normal to γ at $\vec{x}_0$. Furthermore, this is true for every smooth γ lying in S and passing through $\vec{x}_0$. Therefore, $\nabla f(\vec{x}_0)$ must be normal to S.

Example: Find an equation for the tangent plane to the surface $4x^2 + y^2 - z^2 = 4$ at the point $(1, 2, -2)$. Also find an equation for the normal line to this surface at this point. (This is the line through the point that is normal to the surface at that point.)

This is a level surface of the function $f(x, y, z) = 4x^2 + y^2 - z^2$, so a normal vector to the surface is $\nabla f(1, 2, -2)$

$$\nabla f(x, y, z) = \langle 8x, 2y, -2z \rangle \qquad \nabla f(1, 2, -2) = \langle 8, 4, 4 \rangle .$$

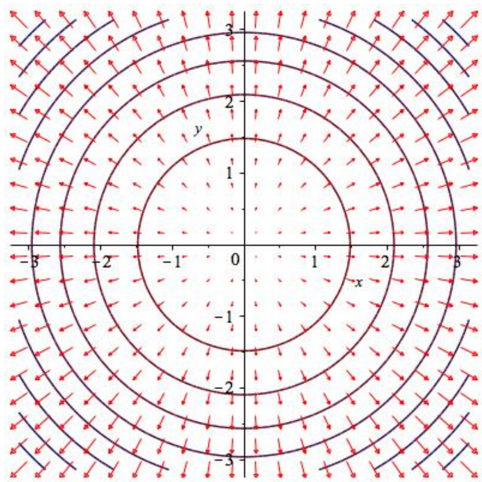
This is a normal vector to the plane, and so is $\langle 2, 1, 1 \rangle$. A point on the plane is $(1, 2, -2)$, so an equation for the plane is

$$2x + y + z = 2.$$

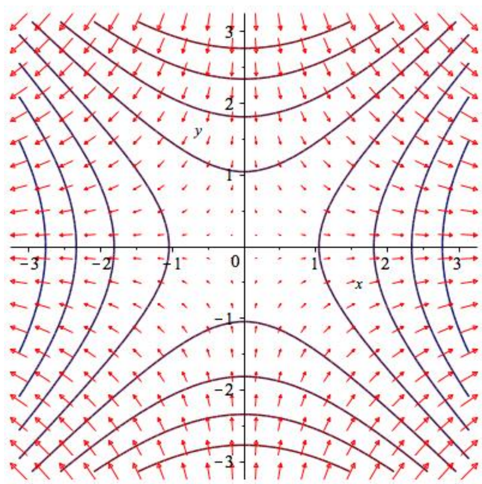
The normal line to the surface at this point is parallel to the normal vector $\langle 2, 1, 1 \rangle$, and so it has equation

$$\langle x, y, z \rangle = \langle 1, 2, -2 \rangle + t \langle 2, 1, 1 \rangle .$$

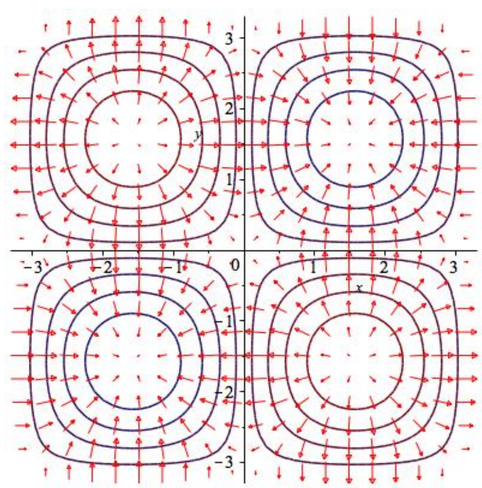
Here are some pictures of the gradient fields of functions $f(x, y)$, together with level curves of those functions. (Note that the arrows are shorter than they should be, but to scale relative to each other.)



$f(x, y) = x^2 + y^2$



$f(x, y) = x^2 - y^2$



$f(x, y) = \sin(x) \sin(y)$

Example: Let $f(x, y) = xe^y$ and $P = (2, 0)$. Find:

1. The directional derivative of f at P in the direction of the vector $\langle 1, -1 \rangle$. (Remember to use a unit vector to represent the direction.)

2. The largest possible directional derivative of f at P.

3. A unit vector in the direction of that largest possible directional derivative.

Example: Find equations for the tangent plane and for the normal line to the ellipsoid $4x^2 + y^2 + 9z^2 = 14$ at the point $(1, 1, 1)$.

Example:

$$f(x,y) = \frac{1}{x^2+y^2+3}$$

1. What is the largest possible value of $D_{\vec{u}}f(3,4)$, and for what unit vector $\vec{u}$ do we have that value?

2. At any particular point (x,y), in which direction $\vec{u}$ do we get the largest possible value of $D_{\vec{u}}f(x,y)$? (Give a geometric answer.) What is that value?

3. What is the largest possible value of $D_{\vec{u}}f(x,y)$ for all possible values of $\vec{u}$, x, and y?

   (Hint: For any particular point (x,y), you just found the largest possible value of $D_{\vec{u}}f(x,y)$. That value is a function of x^2+y^2. Set $t = x^2+y^2$, and now you have a function $g(t)$ to maximize. Second hint: It is easier to maximize $(g(t))^2$.)

   At which points do we have that directional derivative?

Exercise: Prove that if f is differentiable at (x, y) and $\vec{u}$ is a unit vector, the directional derivative of f at (x, y) in the direction $\vec{u}$ is given by

$$\frac{\partial f}{\partial \vec{u}}(x, y) = \nabla f(x, y) \cdot \vec{u}.$$

To prove this, write $\vec{u} = \langle \cos\theta, \sin\theta \rangle$, and set

$$g(h) = f((x, y) + h\vec{u}).$$

First, using the limit definitions of derivative and of directional derivative, show that

$$g'(0) = \frac{\partial f}{\partial \vec{u}}(x, y).$$

Then, using the Chain Rule, show that

$$g'(0) = \nabla f(x, y) \cdot \vec{u}.$$

Mathematical Challenge:

Let $f(x, y)$ be a polynomial of the form

$$f(x, y) = ax^2 + by^2 + 2cxy.$$

Argue that if $ab - c^2 < 0$, then $(x, y) = (0, 0)$ is neither a minimum point nor a maximum point for $f(x, y)$.

(This fact will partially justify the second derivative test, which we will learn next time.)

Suggestion: Consider functions $\vec{g}_m(t) = \langle t, mt, f(t, mt) \rangle$, where m is a constant. Let $h_m(t)$ be the z-coordinate $f(t, mt)$.

First, argue that, if $t = 0$ is a strict[1] minimum point for $h_m(t)$ for some values of m and a strict maximum point for $h_m(t)$ for other values of m, then $(x, y) = (0, 0)$ is neither a minimum point nor a maximum point for $f(x, y)$. (Hint: The function $\vec{g}_m$ parametrizes a curve lying in the graph of f. What does that curve look like? What is its projection on the xy-plane?)

Then, use techniques of single-variable calculus to determine when $t = 0$ is a strict minimum point or a strict maximum point for $h_m(t)$.

[1]To say $t = 0$ is a *strict* minimum point for $h(t)$ means that there are points near $t = 0$ for which $h(t) > h(0)$. Technically, according to the definition (see page 1000 of the textbook), if $h(t)$ is a constant function, then every point is a minimum point for $h(t)$, although no point is a strict minimum point.