Math 11
Fall 2016
Section 1
Wednesday, October 5, 2016

First, some important points from the last class:

Definition: If $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{u} = \langle u_1, \ldots, u_n \rangle$ is a unit vector in $\mathbb{R}^n$, then the *directional derivative* of f at $(x_1, \ldots, x_n)$ in the direction $\vec{u}$ is

$$D_{\vec{u}} f(x_1, \ldots, x_n) = \frac{\partial f}{\partial \vec{u}}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f((x_1, \ldots, x_n) + h(u_1, \ldots, u_n)) - f(x_1, \ldots, x_n)}{h}.$$

This is the rate of change of f with respect to distance, when the argument (input) of f is moving in the direction $\vec{u}$.

If $f : \mathbb{R}^2 \to \mathbb{R}$, then $D_{\vec{u}}(x, y)$ is the slope of the slice of the graph of f in the vertical plane containing the line in the xy-plane through the point (x, y) in the direction of the vector $\vec{u}$.

Theorem: If f is differentiable at $(x_1, \ldots, x_n)$, then

$$D_{\vec{u}}(x_1, \ldots, x_n) = \nabla f(x_1, \ldots, x_n) \cdot \vec{u}.$$

Warning: The vector $\vec{u}$ must be a unit vector.

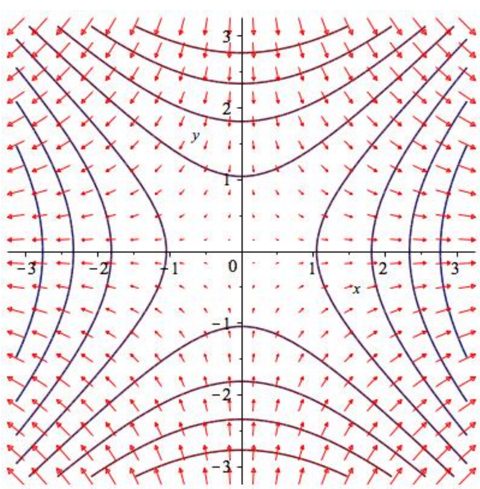
Theorem: If f is differentiable at $(x_1, \ldots, x_n)$ then:

The maximum value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is $|\nabla f(x_1, \ldots, x_n)|$ and it occurs when $\vec{u}$ points in the direction of $\nabla f(x_1, \ldots, x_n)$.

The minimum value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is $-|\nabla f(x_1, \ldots, x_n)|$ and it occurs when $\vec{u}$ points in the opposite direction to $\nabla f(x_1, \ldots, x_n)$.

The value of $D_{\vec{u}} f(x_1, \ldots, x_n)$ is 0 when $\vec{u}$ is perpendicular to $\nabla f(x_1, \ldots, x_n)$.

The vector $\nabla f(x_1, \ldots, x_n)$ is normal to the level set (level curve or level surface) of f containing the point $(x_1, \ldots, x_n)$.



$f(x, y) = x^2 - y^2$

Warm-up Problem: Find all points where the graph of the function

$$f(x, y) = 3x - x^3 - 2y^2 + y^4$$

has a horizontal tangent plane.

This is a polynomial, so it is differentiable everywhere, and it will have a horizontal tangent plane exactly when both partial derivatives equal zero.

$$\frac{\partial f}{\partial x}(x, y) = 3 - 3x^2 = 3(1 - x^2) \qquad \frac{\partial f}{\partial y}(x, y) = -4y + 4y^3 = 4y(y^2 - 1)$$

$$\frac{\partial f}{\partial x}(x, y) = 0 \text{ when } x = 1 \text{ or } x = -1 \qquad \frac{\partial f}{\partial y}(x, y) = 0 \text{ when } y = 0 \text{ or } y = 1 \text{ or } y = -1$$

The tangent plane is horizontal at the points

$$(1, 0) \quad (1, 1) \quad (1, -1) \quad (-1, 0) \quad (-1, 1) \quad (-1, -1).$$

Definition: The point (a, b) is a *critical point* of $f(x, y)$ if either $\nabla f(a, b) = \langle 0, 0\rangle$ or $\nabla f(a, b)$ is undefined.

The point (a, b) is a *local maximum point* of $f(x, y)$ if there is any neighborhood of (a, b) throughout which $f(x, y) \leq f(a, b)$. (A neighborhood of (x, y) is a disc centered at (x, y).)

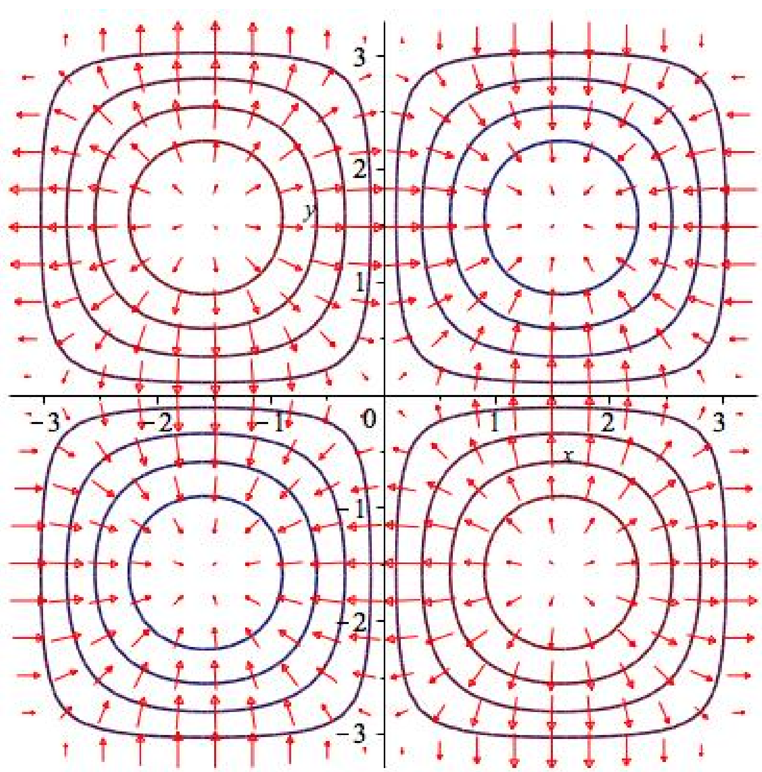
The point (a, b) is a *local minimum point* of $f(x, y)$ if there is any neighborhood of (a, b) throughout which $f(x, y) \geq f(a, b)$.

The point (a, b) is a *saddle point* of $f(x, y)$ if $\nabla f(a, b) = \langle 0, 0\rangle$ and (a, b) is neither a local maximum point nor a local minimum point.

Theorem: Local maximum and minimum points are always critical points.

Note: This applies to functions of more than two variables as well.

Example: Here are some level curves, and the gradient field, of $f(x, y) = \sin(x) \sin(y)$. Where do we see critical points? Are they local maxima, local minima, or saddle points?



Some critical points are found approximately at the origin, and near the centers of the portions of the four quadrants included in this picture.

In the first and third quadrant we have local maximum points, in the second and fourth quadrant we have local minimum points, and at the origin we have a saddle point.

Definition: If (a, b) is a critical point of $f(x, y)$, and all the second partial derivatives of f are defined and continuous near (a, b), we define the discriminant of f at (a, b) to be

$$D(a,b) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{vmatrix}$$

Theorem (the second derivative test): If (a, b) is a critical point of $f(x, y)$, and all the second partial derivatives of f are defined and continuous near (a, b), then

$$D(a,b) > 0 \;\;\&\;\; \frac{\partial^2 f}{\partial x^2}(a,b) < 0 \implies (a,b) \text{ is a local maximum point;}$$

$$D(a,b) > 0 \;\;\&\;\; \frac{\partial^2 f}{\partial x^2}(a,b) > 0 \implies (a,b) \text{ is a local minimum point;}$$

$$D(a,b) < 0 \quad \implies (a,b) \text{ is a saddle point;}$$

$$D(a,b) = 0 \quad \implies \text{the second derivative test fails to give any information about (a,b).}$$

Note: This second derivative test is for functions $f : \mathbb{R}^2 \to \mathbb{R}$. We are not learning a second derivative test for functions $f : \mathbb{R}^3 \to \mathbb{R}$.

Example: Find the critical points of

$$f(x,y) = 2x^3 - x^2y + y$$

and use the second derivative test to classify them as local maximum points, local minimum points, or saddle points.

$$\frac{\partial f}{\partial x}(x,y) = 6x^2 - 2xy = 2x(3x-y) \qquad \frac{\partial f}{\partial y} = -x^2 + 1$$

$$\frac{\partial f}{\partial x}(x,y) = 0 \text{ when } x = 0 \text{ or } y = 3x \qquad \frac{\partial f}{\partial y}(x,y) = 0 \text{ when } x = \pm 1.$$

The critical points are $(1,3)$ and $(-1,-3)$.

$$D(x,y) = \begin{vmatrix} 12x - 2y & -2x \\ -2x & 0 \end{vmatrix} = -4x^2 \qquad D(1,3) = D(-1,-3) = -4 < 0.$$

Both critical points are saddle points.

Example: Find and classify the critical points of

$$f(x,y) = x^4 + y^4.$$

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 \qquad \frac{\partial f}{\partial y} = 4y^3$$

$$\frac{\partial f}{\partial x}(x,y) = 0 \text{ when } x = 0 \qquad \frac{\partial f}{\partial y}(x,y) = 0 \text{ when } y = 0.$$

The critical point is $(0,0)$. At this point,

$$D(x,y) = \begin{vmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

The second derivative test fails. However, we can tell that $f(x,y)$ equals zero at the critical point $(0,0)$ and is positive everywhere else, so $(0,0)$ is a local minimum point.

Example: Find the distance between the lines with vector equations

$$\vec{r} = \langle t+3, t, 12\rangle \qquad \text{and} \qquad \vec{r} = \langle 2t, 9, t+3\rangle .$$

A typical point on the first line is $P_t = \langle t+3, t, 12\rangle$. A typical point on the second line is $Q_s = \langle 2s, 9, s+3\rangle$. The distance between these two points is

$$d_{t,s} = \sqrt{(t-2s+3)^2 + (t-9)^2 + (9-s)^2}.$$

We want to minimize this distance. It's easier to minimize the square of the distance. Set

$$f(t,s) = (t-2s+3)^2 + (t-9)^2 + (9-s)^2.$$

The minimum value of f will be found at a critical point.

$$f_t(t,s) = 2(t-2s+3)+2(t-9) = 4t-4s-12; \quad f_s(t,s) = -4(t-2s+3)-2(9-s) = -4t+10s-30.$$

The critical point occurs when $4t - 4s - 12 = 0$ and $-4t + 10s - 30 = 0$, or when $(t,s) = (10,7)$. The minimum value of $f(t,s)$ is

$$f(10,7) = (10-14+3)^2 + (10-9)^2 + (2)^2 = 6.$$

Since $f(t,s)$ is the square of the distance between points P_t and Q_s, the distance between the lines is $\sqrt{6}$.

To check this answer: Set $P = P_{10}$ and $Q = Q_7$. Check that $|PQ| = \sqrt{6}$ and that $\overrightarrow{PQ}$ is perpendicular to both lines.

Question: How do we know f has a minimum value? Does it have a maximum value?

Example: The only critical point of $f(x, y) = x^2 + y^2$ is $(0, 0)$, which is a local minimum point.

What are the largest and smallest values of $f(x, y)$ on the square region D defined by $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, and where are they located?

Since $f(x, y)$ is the square of the distance from the origin to (x, y), we can analyze this one without too much trouble.

The smallest value is $f(0, 0) = 0$.

The largest value is $f(1, 1) = f(1, -1) = f(-1, 1) = f(-1, -1) = 2$.

The smallest value occurs inside D, at a critical point of f. The largest value occurs at the edge of D. This illustrates the general case.

Definition: A region D is *bounded* if there is some number b such that every point in D has a distance from the origin of at most b.

D is *open* if every point that belongs to D has a neighborhood that is included in D.

D is *closed* if every edge point of D belongs to D. (In three dimensions, every point on the surface of D belongs to D.)

Example: The region $x^2 + y^2 < 1$ is open and bounded.

The region $x^2 + y^2 \leq 1$ is closed and bounded.

The region $1 < x^2 + y^2 \leq 4$ is bounded, and neither closed nor open.

The region $x^2 + y^2 \geq 1$ is closed and unbounded.

Definition: The number c is an *absolute maximum* value for $f(x, y)$ on D if there is some point (a, b) in D such that $f(a, b) = c$, and for all points (x, y) in D we have $f(x, y) \leq c$. The absolute maximum value c is attained at (a, b).

The number c is an *absolute minimum* value for $f(x, y)$ on D if there is some point (a, b) in D such that $f(a, b) = c$, and for all points (x, y) in D we have $f(x, y) \geq c$. The absolute minimum value c is attained at (a, b).

Theorem: A continuous function $f(x, y)$ defined on a closed bounded region D has an absolute maximum value and an absolute minimum value on D. The points at which those extreme values are attained are either critical points of f or edge points of D.

Note: The single-variable version of this theorem is: A continuous function $f(x)$ on a closed (bounded) interval $[a, b]$ has an absolute maximum value and an absolute minimum value on $[a, b]$. The points at which those extreme values are attained are either critical points of f or end points of $[a, b]$.

Example: Find the largest and smallest values of $f(x,y) = x^2 - y^2$ on the region $x^2 + y^2 \leq 1$.

There is one critical point of f, the origin $(0,0)$, and

$$f(0,0) = 0.$$

This is a possible candidate for the largest or smallest value.
Now we have to check the edge points.

Method 1: Parametrize the edge, by $(x,y) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$, and find the largest and smallest values of $f(\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$.

$$g(t) = f(\cos(t), \sin(t)) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$$
$$g'(t) = -4\sin(t)\cos(t)$$

Check critical points of g and end points of the interval. End points: $t = 0$ $((x,y) = (1,0))$, $t = 2\pi$ $((x,y) = (1,0))$. Critical points other than end points: When $\sin(t) = 0$, $t = \pi$, $(x,y) = (-1,0)$. When $\cos(t) = 0$, $t = \dfrac{\pi}{2}$ $((x,y) = (0,1))$, $t = \dfrac{3\pi}{2}$ $((x,y) = (0,-1))$. This gives these possible candidates for maximum or minimum value of f:

$$f(1,0) = 1 \qquad f(-1,0) = 1 \qquad f(0,1) = -1 \qquad f(0,-1) = -1.$$

Compare these to $f(0,0) = 0$.
The maximum value is $f(1,0) = f(-1,0) = 1$, and the minimum value is $f(0,1) = f(0,-1) = -1$.

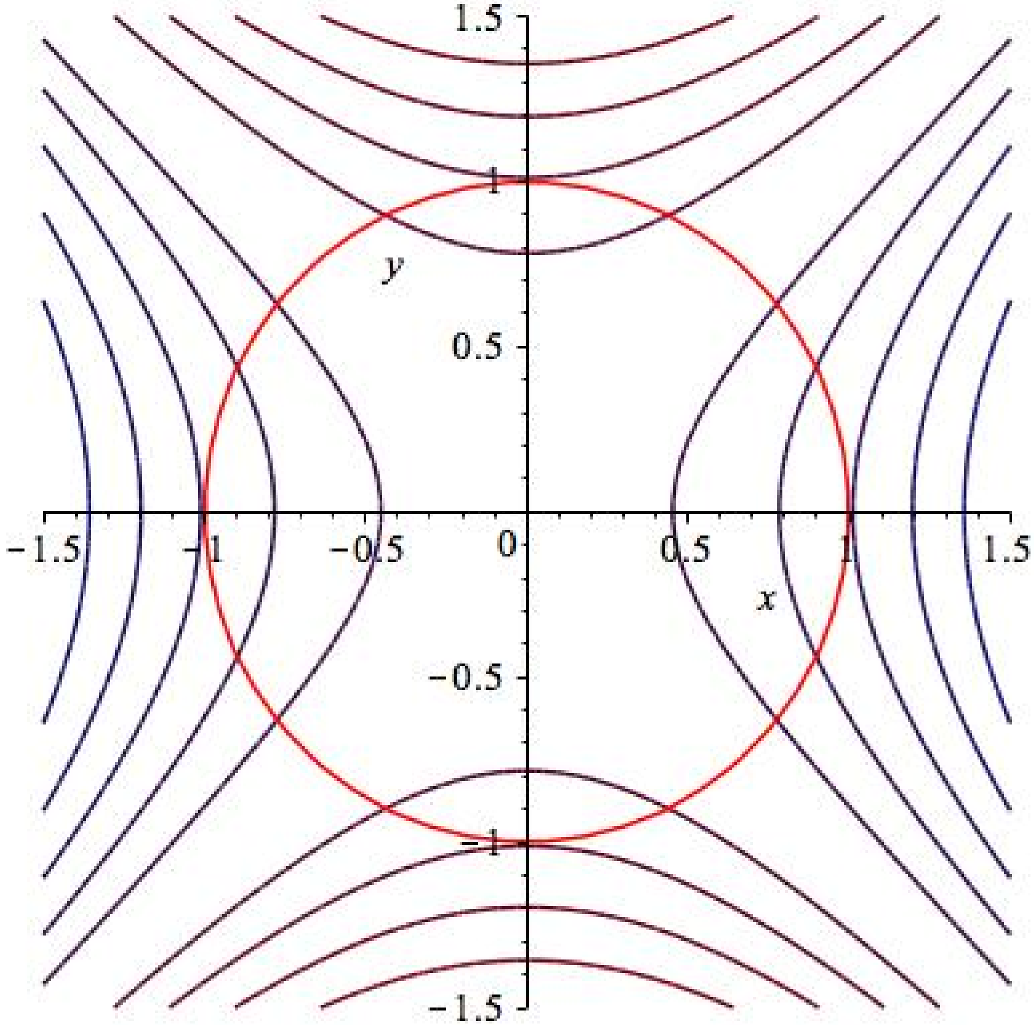
Method 2: Write y in terms of x on the top half of the circle:

$$y = \sqrt{1-x^2} \qquad -1 \leq x \leq 1 \quad f(x,y) = f(x, \sqrt{1-x^2}) = x^2 - (1-x^2) = 2x^2 - 1 = h(x).$$

Now find the largest and smallest values of $h(x)$ by checking critical points and end points.
Critical point: $h'(x) = 4x = 0$ when $x = 0$ $((x,y) = (0,1))$.
End points: $x = -1$ $((x,y) = (-1,0))$, $x = 1$ $((x,y) = (1,0))$.
This gives $(0,1)$, $(-1,0)$,and $(1,0)$ as candidate edge points at which f could reach its maximum or minimum value.
Doing the same on the bottom half of the circle gives $(0,-1)$, $(-1,0)$,and $(1,0)$.
Now we have the same five points to check as before:

$$f(1,0) = 1 \qquad f(-1,0) = 1 \qquad f(0,1) = -1 \qquad f(0,-1) = -1 \qquad f(0,0) = 0.$$

Method 3: We'll see another way to check edge points next class.



The edge of the region D, and some level curves of f, from the preceding page.

Example: Find all critical points of the function

$$f(x, y) = x^3 + 3xy + y^2 + 2y$$

and classify each of them as a local maximum point, local minimum point, or saddle point.

Example: Find the largest and smallest values of the function $f(x, y) = 3x^2 - y$ on the region $x^2 \leq y \leq 1$. (Hint: Draw this region first. Reminder: You must check critical points and edge points.)

Example: Find the volume of the largest rectangular box with sides aligned with the coordinate planes, located in the first octant, with one corner at the origin and the other on the downward-facing elliptical paraboloid $z = 16 - 4x^2 - y^2$.