Math 11
Fall 2016
Section 1
Friday, October 7, 2016

First, some important points from the last class:

Definition: The point (a, b) is a *critical point* of $f(x, y)$ if either $\nabla f(a, b) = \langle 0, 0\rangle$ or $\nabla f(a, b)$ is undefined.

The point (a, b) is a *local maximum point* of $f(x, y)$ if there is any neighborhood of (a, b) throughout which $f(x, y) \leq f(a, b)$.

The point (a, b) is a *local minimum point* of $f(x, y)$ if there is any neighborhood of (a, b) throughout which $f(x, y) \geq f(a, b)$.

The point (a, b) is a *saddle point* of $f(x, y)$ if $\nabla f(a, b) = \langle 0, 0\rangle$ and (a, b) is neither a local maximum point nor a local minimum point.

Theorem: Local maximum and minimum points are always critical points.

Definition: If (a, b) is a critical point of $f(x, y)$, and all the second partial derivatives of f are defined and continuous near (a, b), we define the discriminant of f at (a, b) to be

$$D(a, b) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(a, b) & \frac{\partial^2 f}{\partial x \partial y}(a, b) \\ \\ \frac{\partial^2 f}{\partial y \partial x}(a, b) & \frac{\partial^2 f}{\partial y^2}(a, b) \end{vmatrix}$$

Theorem (the second derivative test): If (a, b) is a critical point of $f(x, y)$, and all the second partial derivatives of f are defined and continuous near (a, b), then

$$D(a, b) > 0 \;\; \& \;\; \frac{\partial^2 f}{\partial x^2}(a, b) < 0 \implies (a, b) \text{ is a local maximum point;}$$

$$D(a, b) > 0 \;\; \& \;\; \frac{\partial^2 f}{\partial x^2}(a, b) > 0 \implies (a, b) \text{ is a local minimum point;}$$

$$D(a, b) < 0 \quad \implies (a, b) \text{ is a saddle point;}$$

$$D(a, b) = 0 \quad \implies \text{ the second derivative test fails to give any information about (a,b).}$$

Definition: A region D is *bounded* if there is some number b such that every point in D has a distance from the origin of at most b.

D is *open* if every point that belongs to D has a neighborhood that is included in D.

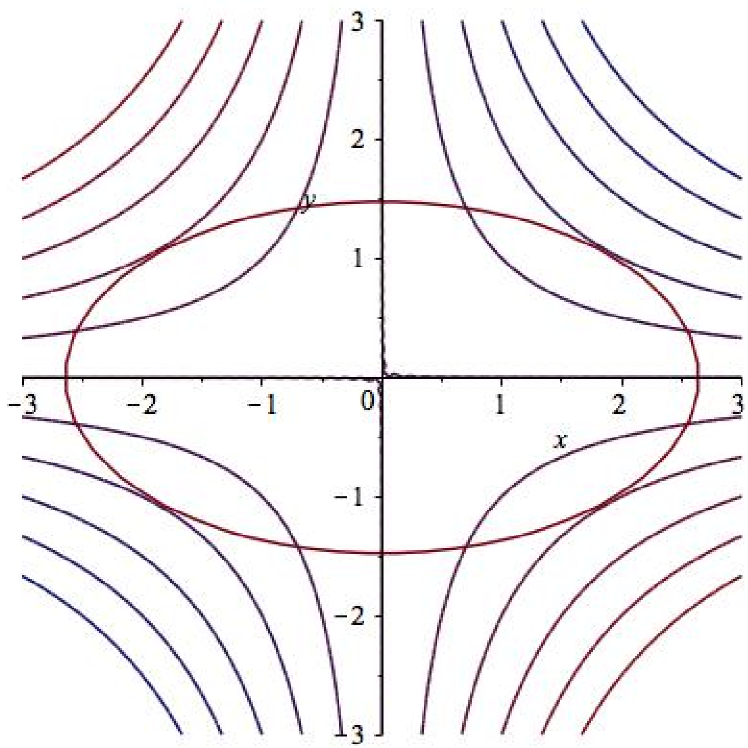
D is *closed* if every edge point of D belongs to D. (In three dimensions, every point on the surface of D belongs to D.)

Definition: The number c is an *absolute maximum* value for $f(x, y)$ on D if there is some point (a, b) in D such that $f(a, b) = c$, and for all points (x, y) in D we have $f(x, y) \leq c$. The absolute maximum value c is attained at (a, b).

The number c is an *absolute minimum* value for $f(x, y)$ on D if there is some point (a, b) in D such that $f(a, b) = c$, and for all points (x, y) in D we have $f(x, y) \geq c$. The absolute minimum value c is attained at (a, b).

Theorem: A continuous function $f(x, y)$ defined on a closed bounded region D has an absolute maximum value and an absolute minimum value on D. The points at which those extreme values are attained are either critical points of f or edge points of D.

Warm-up Problem: Pictured are some level curves of the function $f(x, y) = xy$, and an ellipse γ, which is a level curve of a function $g(x, y)$.



1. Give the approximate coordinates of the points on γ at which $f(x, y)$ is largest and smallest.

2. What relationship do the level curves of f and of g have at those points?

3. What relationship do the gradients of f and of g have at those points? Why?

We can see the largest and smallest values of $f(x, y)$ on γ are at approximately $(\pm 2, \pm 1)$. At those points the ellipse is tangent to the level curve of $f(x, y)$. Because the ellipse is a level curve of $g(x, y)$, and gradients are normal to level curves, this means the gradients of f and g must be parallel at those points.

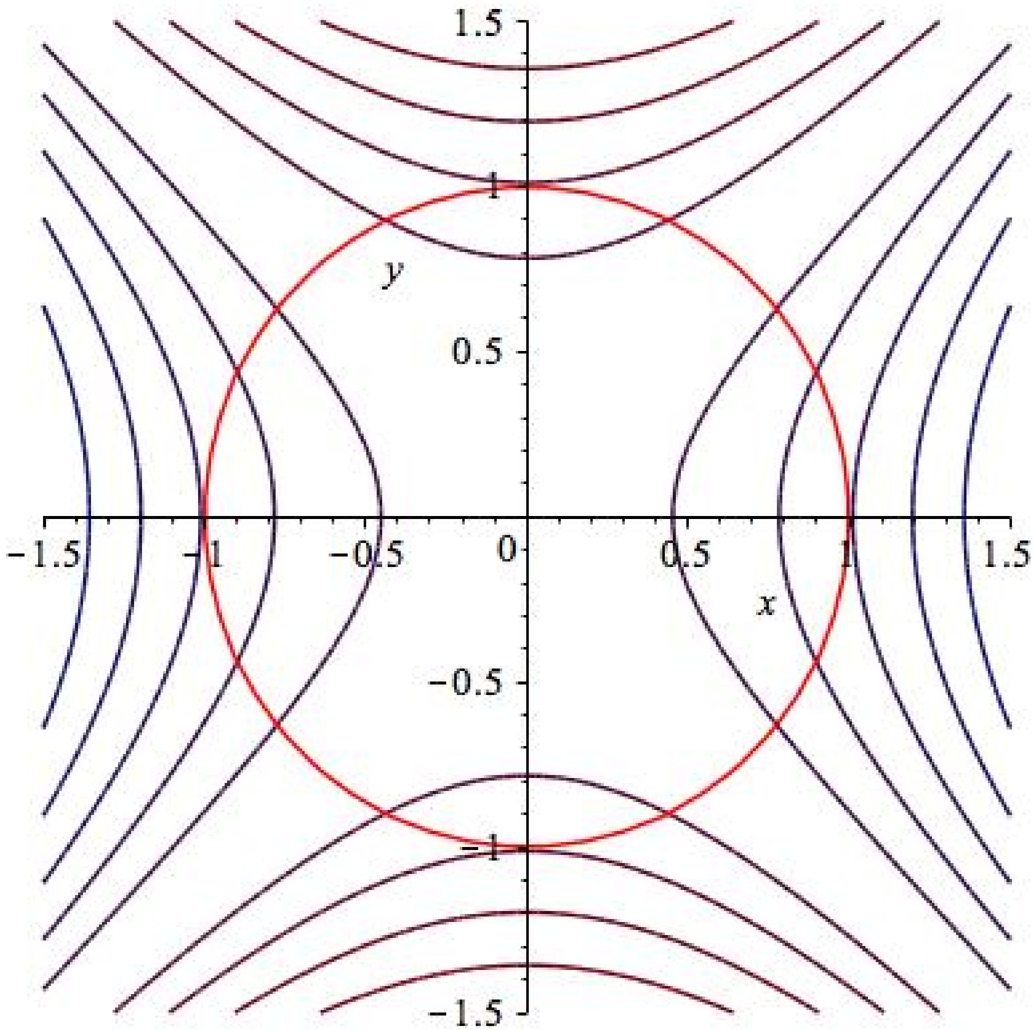
From last class: Find the largest and smallest values of $f(x, y) = x^2 - y^2$ on the region $x^2 + y^2 \leq 1$.

There is one critical point of f, the origin $(0, 0)$, and

$$f(0, 0) = 0.$$

This is a possible candidate for the largest or smallest value.

Now we have to check the edge points.



The edge, the circle $x^2 + y^2 = 1$, is a level curve of $g(x, y) = x^2 + y^2$. We look for points at which the circle is tangent to a level curve of $f(x, y) = x^2 - y^2$ by looking for points on the circle at which ∇f and ∇g are parallel. Parallel vectors are scalar multiples of each other. So we want to find points (x, y) (and some scalar λ) satisfying:

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$g(x, y) = 1.$$

For our example, these equations become

$$\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$$

$$x^2 + y^2 = 1.$$

We can break up our first equation into its x and y components:

$$2x = \lambda 2x$$

$$-2y = \lambda 2y$$

$$x^2 + y^2 = 1.$$

Our first equation gives us $x = 0$ or $\lambda = 1$.

For $x = 0$, our third equation gives us $y = \pm 1$.

For $\lambda = 1$, our second equation gives us $y = 0$, so our third equation gives us $x = \pm 1$.

This gives the four points $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$ to add to $(0, 0)$ as possible locations of the largest and smallest value of $f(x, y)$ on the region $x^2 + y^2 \leq 1$. Evaluating $f(x, y)$ at these points, we see the extreme values of $f(x, y)$ on our region are

$$f(1, 0) = f(-1, 0) = 1 \qquad f(0, 1) = f(0, -1) = -1.$$

Finding the largest or smallest value of $f(x_1, \ldots, x_1)$ is called an optimization problem. Finding the largest or smallest value of $f(x_1, \ldots, x_n)$ when $(x_1, \ldots, x_n)$ is required to satisfy some condition (for example, $x^2 + y^2 = 1$) is called a constrained optimization problem, and the condition is the constraint.

When we are trying to maximize or minimize f on a closed, bounded, region, looking at the edge of that region generally involves constraints of the form $g(x_1, \ldots, x_n) = k$ (for example, $x^2 + y^2 = 1$). In other words, $(x_1, \ldots, x_n)$ must lie on some level set (level curve, level surface, $\ldots$) of g.

The method we just used, called the method of Lagrange multipliers, is designed to solve exactly this kind of problem.

Theorem (the method of Lagrange multipliers): Suppose $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ are differentiable functions, and S is a level set of g, defined by $g(x_1, \ldots, x_n) = k$.

If $f(x_1, \ldots, x_n)$ has a largest (or smallest) value on S, then it attains that extreme value at a point $(x_1, \ldots, x_n)$ at which either

$$\nabla g(x_1, \ldots, x_n) = \vec{0}$$

or, for some scalar λ,

$$\nabla f(x_1, \ldots x_n) = \lambda \nabla g(x_1, \ldots, x_n).$$

This means that to solve this problem, we should look for solutions to

$$\nabla g(x_1, \ldots, x_n) = \vec{0} \quad \& \quad g(x_1, \ldots, x_n) = k$$

and to

$$\nabla f(x_1, \ldots, x_n) = \lambda \nabla g(x_1, \ldots, x_n) \quad \& \quad g(x_1, \ldots, x_n) = k.$$

Example: Find the distance between the plane $x+2y-z=16$ and the point $P=(3,1,1)$.

To do this, we find the point on the plane that is closest to P. The distance between any point (x,y,z) and the point P is

$$d=\sqrt{(x-3)^2+(y-1)^2+(z-1)^2},$$

and we want to find the smallest value of d on the plane. It is easier to find the smallest value of d^2, so we define

$$f(x,y,z)=(x-3)^2+(y-1)^2+(z-1)^2.$$

We want to find the smallest value of $f(x,y,z)$ on the surface $x+2y-z=0$, which is a level surface of the function $g(x,y,z)=x+2y-z$.

Since $\nabla g(x,y,z)=\langle 1,2,-1\rangle$, we know $\nabla g(x,y,z)$ is never zero, so we must solve:

$$\nabla f(x_1,\ldots,x_n)=\lambda\nabla g(x_1,\ldots,x_n) \qquad \& \qquad g(x_1,\ldots,x_n)=k.$$

$$\langle 2(x-3),2(y-1),2(z-1)\rangle=\lambda\langle 1,2,-1\rangle \qquad \& \qquad x+2y-z=16$$

Breaking up the first equation into its x, y, and z components gives

$$2x-6=\lambda \qquad 2y-2=2\lambda \qquad 2z-2=-\lambda \qquad x+2y-z=16.$$

These are four linear equations in four unknowns, so we can solve this system:

$$\lambda=2x-6$$

$$2y-2=2\lambda \implies 2y-2=2(2x-6) \implies y=2x-5$$

$$2z-2=-\lambda \implies 2z-2=-(2x-6) \implies z=4-x$$

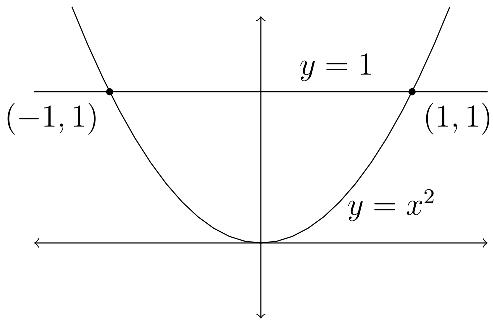
$$x+2y-z=16 \implies x+2(2x-5)-(4-x)=16 \implies x=5$$

$$x=5 \qquad y=2x-5=5 \qquad z=4-x=-1$$

The closest point on the plane to P is $(5,5,-1)$, so the distance between the plane and $P=(3,1,1)$ is the distance between these two points,

$$\sqrt{(5-3)^2+(5-1)^2+(-1-1)^2}=\sqrt{2^2+4^2+2^2}=\sqrt{24}=2\sqrt{6}.$$

Example: Find the largest and smallest values of the function $f(x, y) = x^2 + xy$ on the region $x^2 \leq y \leq 1$.



$\nabla f(x, y) = \langle 2x + y, x \rangle$, and the only critical point is $(0, 0)$. This gives $(0, 0)$ as a possible maximum or minimum point. Now we must look at the edge of the region, which consists of a line segment and part of a parabola.

The parabola is a level curve of $g(x, y) = x^2 - y$, namely $g(x, y) = 0$. $\nabla g(x, y) = \langle 2x, -1 \rangle$, which is never zero. Using Lagrange multipliers to find possible largest and smallest values of f on the parabola, we must solve

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \& \qquad g(x, y) = 0$$

$$\langle 2x + y, x \rangle = \lambda \langle 2x, -1 \rangle \qquad \& \qquad x^2 - y = 0$$

which yields the system of equations

$$2x + y = \lambda 2x \qquad x = -\lambda \qquad y = x^2,$$

which has solutions $x = 0, y = 0, \lambda = 0$ and $x = -\frac{2}{3}, y = \frac{4}{9}, \lambda = \frac{2}{3}$. This gives $(0, 0)$ and $\left(-\frac{2}{3}, \frac{4}{9}\right)$ as possible maximum or minimum points. Since we are only looking at a segment of the parabola, we must also check the end points $(-1, 1)$ and $(1, 1)$.

On the line segment $y = 1$, we have $f(x, y) = x^2 + x$, which we can check has a critical point at $x = -\frac{1}{2}$, giving $\left(-\frac{1}{2}, 1\right)$ as a possible maximum or minimum point. We must also check the end points, but they are already on our list.

$$f(0, 0) = 0 \qquad f(-1, 1) = 2 \qquad f(1, 1) = 2 \qquad f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4} \qquad f\left(-\frac{2}{3}, \frac{4}{9}\right) = \frac{4}{27}$$

We see that the maximum value of 2 is attained at $(-1, 1)$ and at $(1, 1)$, and the minimum value of $-\frac{1}{4}$ is attained at $\left(-\frac{1}{2}, 1\right)$.

Example: Find the largest and smallest values of the function $f(x, y, z) = x^2 - y^2 + z^2$ on the region $x^2 + 4y^2 + 9z^2 \leq 36$.

Remember to check critical points of f inside the region.

Then use the method of Lagrange multipliers to look for maximum and minimum points on the surface of the region.

Example: If possible, find the largest value of the function $f(x, y) = x^2 + 4y^2$ on the branch of the hyperbola $xy = 1$ in the first quadrant. If this is not possible, explain why.

Now do the same thing for the smallest value of f.

Example: Use the method of Lagrange multipliers to find the volume of the largest rectangular box with sides aligned with the coordinate planes, located in the first octant, with one corner at the origin and the opposite corner on the downward-facing elliptical paraboloid $z = 16 - 4x^2 - y^2$. (Hint: The volume of the box is a function of the coordinates of the corner point opposite the origin.)