Math 11
Fall 2016
Section 1
Wednesday, October 12, 2016

First, some important points from the last class:

Definition: If

$$R = \{(x, y) \mid a \le x \le b \;\&\; c \le y \le d\} = [a, b] \times [c, d]$$

and the domain of $f(x, y)$ includes R, then

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A.$$

The intervals $[a, b]$ and $[c, d]$ are divided into m subintervals of length $\Delta x = \dfrac{b-a}{m}$ and n subintervals of length $\Delta y = \dfrac{d-c}{n}$, respectively. This divides R into mn-many rectangles of area $\Delta A = \Delta x \Delta y$.

The point (x_{ij}^*, y_{ij}^*) can be any point in rectangle ij, which corresponds to the i^{th} subinterval of $[a, b]$ and the j^{th} subinterval of $[c, d]$.

$$\lim_{m,n\to\infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \right) = V$$

means that for every $\varepsilon > 0$ [output accuracy] there is a number N [input accuracy] such that whenever $m, n > N$, no matter how the points (x_{ij}^*, y_{ij}^*) are chosen, we have

$$\left| \left(\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \right) - V \right| < \varepsilon.$$

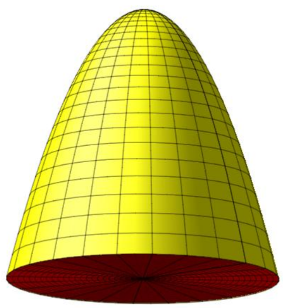
If $\iint_R f(x, y)\, dA$ exists, we say that f is *integrable* over the rectangle R.

Fubini's Theorem: If f is continuous on $R = [a, b] \times [c, d]$ then f is integrable on R and

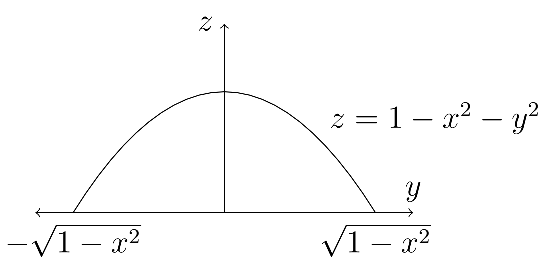
$$\iint_R f(x, y)\, dA = \overbrace{\int_a^b \underbrace{\int_c^d f(x, y)\, dy}_{c,d \text{ limits on } y}\, dx}^{a,b \text{ limits on } x} = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

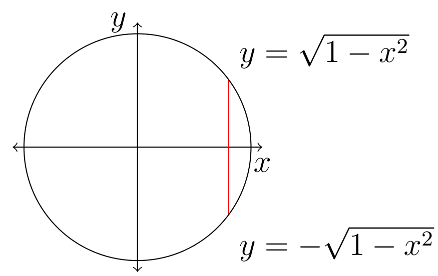
The average value of $f(x, y)$ on R is given by $\dfrac{\iint_R f(x, y)\, dA}{\text{area(R)}}$.

Example: Find the volume of the region above the xy plane and under the paraboloid $z = 1 - x^2 - y^2$.



The region is shown above; the top surface is the part of the paraboloid above the xy plane, and the bottom surface is the unit disc $x^2 + y^2 \leq 1$ in the xy plane. We can use volumes by slicing. Here is a typical slice perpendicular to the x-axis (so x is a constant), together with a view of the object's base indicating where that slice is taken.





The cross-sectional area is

$$A(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy = \left(y(1-x^2) - \frac{y^3}{3} \right) \Bigg|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} = \frac{10}{3}(1-x^2)^{\frac{3}{2}}$$

and the volume is

$$V = \int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \int_{-1}^{1} \left(\frac{10}{3}(1-x^2)^{\frac{3}{2}} \right) dx.$$

If D is the unit disc, the volume of the region above D and below $z = 1 - x^2 - y^2$ is

$$V = \iint_D (1 - x^2 - y^2)\, dA.$$

Double integrals over general regions:

If D is any bounded region in the xy plane, and $f(x,y)$ is a function defined on D, we formally define

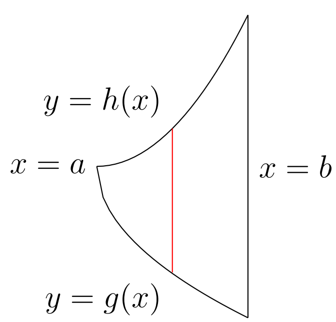
$$\iint_D f(x,y)\,dA = \iint_R g(x,y)\,dA,$$

where R is any rectangle containing D, and g is defined by

$$g(x,y) = \begin{cases} f(x,y) & (x,y) \in D; \\ \\ 0 & (x,y) \notin D. \end{cases}$$

We compute D using the idea of volumes by slicing:

Type I Region R: $a \le x \le b$, $g(x) \le y \le h(x)$:



Notice that the red line shows the limits on y for a fixed value of x.

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx.$$

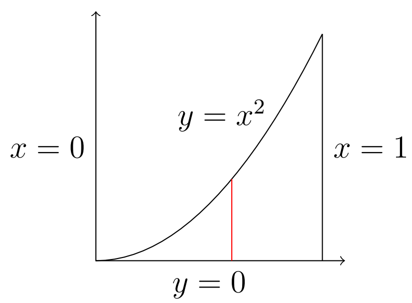
Note that the limits on x are constants, and the limits on y are functions of x.

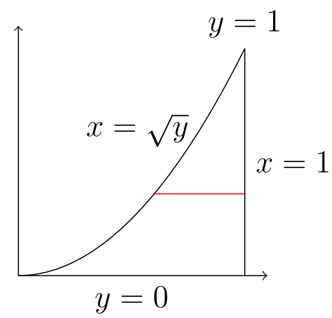
Type II Region R: $a \le y \le b$, $g(y) \le x \le h(y)$:

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy.$$

Example: If R is the region above the x-axis, under the curve $y = x^2$ and between the lines $x = 0$ and $x = 1$, and $f(x, y) = \cos(x^3)$, write $\iint_R f(x, y)\, dA$ using both orders of integration. Evaluate the integral.

We look at the region as both a Type I region and a Type II region:





$$\int_0^1 \int_0^{x^2} \cos(x^3)\, dy\, dx = \iint_R f(x, y)\, dA = \int_0^1 \int_{\sqrt{y}}^1 \cos(x^3)\, dx\, dy.$$

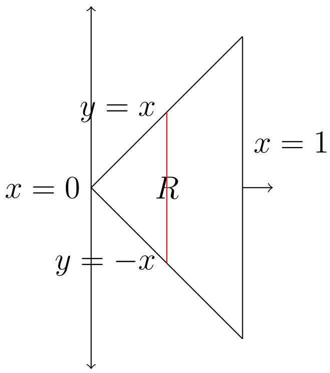
Which shall we evaluate? Let's try the left-hand one:

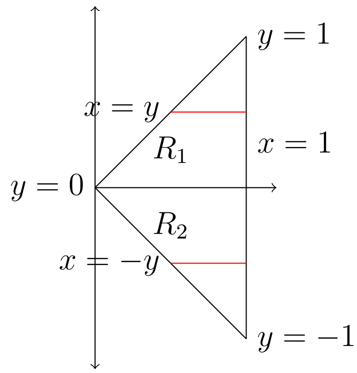
$$\int_0^{x^2} \cos(x^3)\, dy = y \cos(x^3)\Big|_{y=0}^{y=x^2} = x^2 \cos(x^3)$$

$$\int_0^1 \int_0^{x^2} \cos(x^3)\, dy\, dx = \int_0^1 x^2 \cos(x^3)\, dx = \frac{\sin(x^3)}{3}\Big|_{x=0}^{x=1} = \frac{\sin(1)}{3}.$$

Example: Rewrite the given integral as an integral or sum of integrals in the opposite order. Then evaluate it. Try to use symmetry rather than actually antidifferentiating anything.

$$\int_0^1 \int_{-x}^{x} x^2 y \, dy \, dx.$$





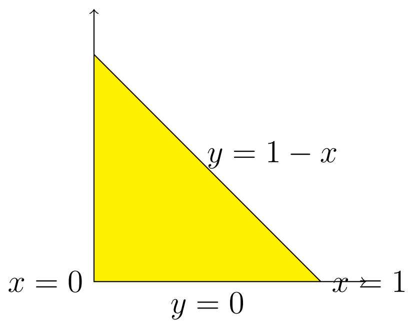
$$\iint_R x^2 \, dA = \iint_{R_1} x^2 y \, dA + \iint_{R_2} x^2 y \, dA = \int_0^1 \int_y^1 x^2 y \, dx \, dy + \int_{-1}^0 \int_{-y}^1 x^2 y \, dx \, dy$$

To evaluate the integral using symmetry, note that $x^2 y$ is an odd function of y. Therefore, since the region of integration is symmetric about the x-axis $y = 0$, volume above the xy plane equals volume below the xy plane, so the value of the integral is 0.

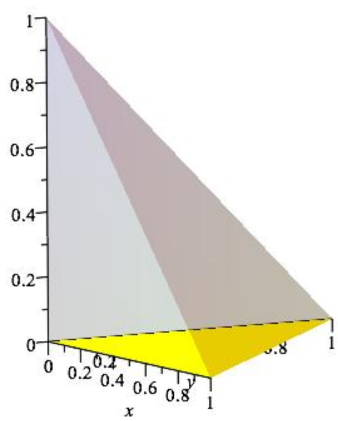
Example: Sketch the three-dimensional region whose volume is given by

$$\int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx.$$

First sketch the region in the xy plane over which we are integrating:

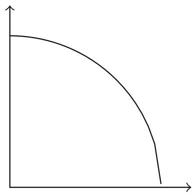


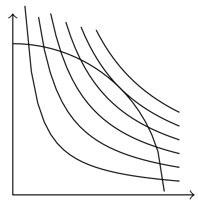
Now consider the surface $z = 1 - x - y$, which can be rewritten $x + y + z = 1$. This is a plane containing the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. With this information we can sketch our region of integration D, the lower boundary of our three-dimensional region, and the surface $z = 1 - x - y$ above D, the upper boundary of our three-dimensional region.



This is the corner of the first octant cut off by the plane $x + y + z = 1$.

Example: Estimate the value of $\iint_D xy\,dA$ where D is the region $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 1$.





Find the maximum and minimum values of xy on D. Clearly, the minimum value is 0, and a look at the contour plot of xy tells us the maximum value must be at the point on the portion of the unit circle bounding D at which $x = y$, namely $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Since $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2}$, on D we have

$$0 \leq xy \leq \frac{1}{2};$$

$$\iint_D 0\,dA \leq \iint_D xy\,dA \leq \iint_D \frac{1}{2}\,dA;$$

$$0 \leq \iint_D xy\,dA \leq \frac{1}{2}(\text{area(D)}) = \frac{\pi}{8}.$$

We could get a better approximation by dividing D into several regions and applying this reasoning to each separately.

This is like approximating an integral $\int_a^b f(x)\,dx$ by the upper and lower Riemann sums. For the upper sum, we choose from interval i a point x_i^* at which f reaches a maximum on interval i, and for the lower sum, we choose from interval i a point x_i^* at which f reaches a minimum on interval i. Then we have

$$\text{lower sum } \leq \int_a^b f(x)\,dx \leq \text{ upper sum}.$$

Furthermore, we can make the approximation as close as we want, by dividing finely enough.

Example: Find an expression, involving an iterated integral, for the average distance from the origin of a point in the unit disc.

$$\frac{1}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2 + y^2} \, dy \, dx$$

Example: Find the value of

$$\iint_D (x + 2y - xy + 4)\, dA,$$

where D is the unit disc, using symmetry and geometric arguments.

Hint: $\displaystyle \iint_R f(x,y) + g(x,y)\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA.$

Example: Write down a double integral representing

$$\iint_R (x^2 - y^2)\, dA,$$

where R is the region given by $0 \leq x \leq 1$ and $x^2 - y^2 \geq 0$. Then evaluate the integral.

Example: Write down iterated integrals representing the volume of the three-dimensional region given by $x^2 + y^2 \leq 4$, $y^2 + z^2 \leq 4$, and $z \geq 0$, in both orders of integration. Then find the volume by evaluating one of the integrals.

Example: A hemispherical bowl of radius a contains liquid with maximum depth h. Write down an iterated integral representing the volume of the liquid in the bowl.