Math 11
Fall 2016
Section 1
Friday, October 14, 2016

First, some important points from the last class:

Double integrals over general regions:

If D is any bounded region in the xy plane, and $f(x, y)$ is a function defined on D, we formally define

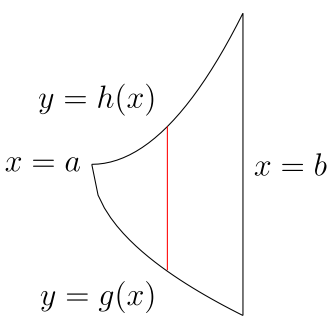
$$\iint_D f(x,y)\,dA = \iint_R g(x,y)\,dA,$$

where R is any rectangle containing D, and g is defined by

$$g(x,y) = \begin{cases} f(x,y) & (x,y) \in D; \\ \\ 0 & (x,y) \notin D. \end{cases}$$

We compute D using the idea of volumes by slicing:

Type I Region R: $a \le x \le b$, $g(x) \le y \le h(x)$:



The red line shows the limits on y for a fixed value of x.

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx.$$

The limits on x are constants, and the limits on y are functions of x.

Type II Region R: $a \le y \le b$, $g(y) \le x \le h(y)$:

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy.$$

Triple Integrals:

Everything that we did with double integrals transfers pretty much wholesale to triple integrals. We define the triple integral over a box (a rectangular parallelepiped) as a limit of Riemann sums, and use this to define the triple integral over a general region. We have the three-dimensional version of Fubini's Theorem:

To evaluate a triple integral
$$\iiint_D f(x,y,z)\,dV,$$
the integral (with respect to volume) of f over the three-dimensional region D, we use an iterated integral:
$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

We can think of this in two ways:
$$\underbrace{\int_a^b}_{(x \text{ limits on entire region})} \underbrace{\int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)}}_{(y,z \text{ limits for fixed } x)} f(x,y,z)\,dz\,dy\,dx.$$

$$\underbrace{\int_a^b \int_{g_1(x)}^{g_2(x)}}_{(x,y \text{ limits on entire region})} \underbrace{\int_{h_1(x,y)}^{h_2(x,y)}}_{(z \text{ limits for fixed } x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Being able to set up double integrals helps us set up triple integrals.

If we are thinking the first way, the inner limits on y and z are the limits of a double integral over the cross-section at fixed x, viewed as a region in the yz plane.

If we are thinking the second way, the outer limits on x and y are the limits of a double integral over the projection of D onto the xy plane.

In any case, in the given order of integration, we always have the following:

The outer limits are numbers. They are the largest and smallest values of x over the entire region.

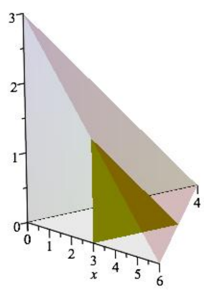
The intermediate limits are functions of x. They are the largest and smallest values of y over the entire cross-section at some fixed x.

The inner limits are functions of x and y. They are the largest and smallest values of z for some fixed x and y.

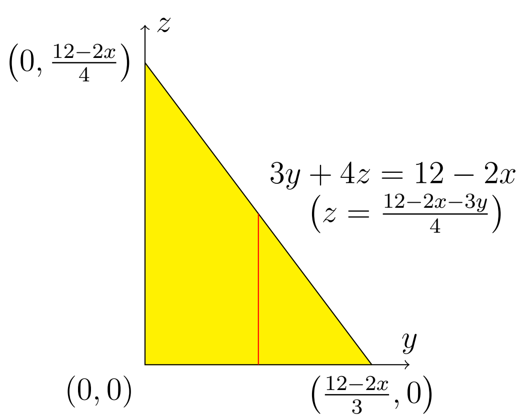
Example: Express $\iiint_D f(x,y,z)\,dA$ as an iterated integral, where D is the corner of the first octant cut off by the plane $2x+3y+4z=12$.

Note that this plane contains the three points $(6,0,0)$, $(0,4,0)$, $(0,0,3)$.

First we think about setting up the integral in the first way on the previous page. Our first picture shows our region together with a typical cross-section for fixed x.



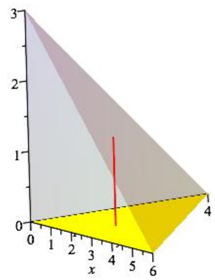
From the picture we see that the limits on x on the entire region are $0 \le x \le 6$, and the cross-section at x looks like a triangle in the yz plane whose sides are the y-axis, the z-axis, and the line $3y+4z=12-2x$. The next picture shows how we set up the y and z limits over this cross-section: $0 \le y \le \dfrac{12-2x}{3}$, and for fixed y we have $0 \le z \le \dfrac{12-2x-3y}{4}$.



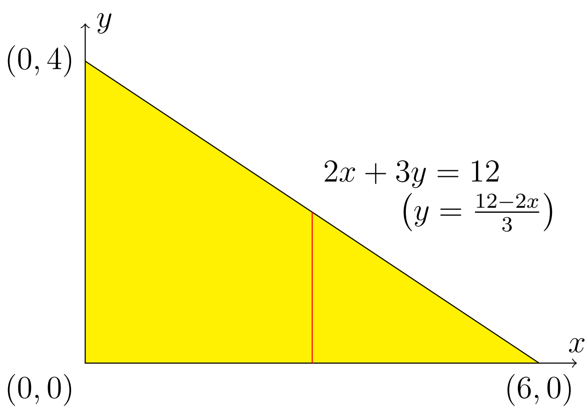
Therefore our integral becomes

$$\int_0^6 \int_0^{\frac{12-2x}{3}} \int_0^{\frac{12-2x-3y}{4}} f(x,y,z)\,dz\,dy\,dx.$$

Now we think about setting up the integral in the second way. Our first picture shows our region together with the projection on the xy plane, and a vertical red line showing the limits on z for fixed x and y. The bottom surface is the plane $z = 0$, and the top surface is the plane $2x + 3y + 4z = 12$, or $z = \dfrac{12 - 2x - 3y}{4}$.



From the picture we see that the limits on z for fixed x and y are $0 \le z \le \dfrac{12 - 2x - 3y}{4}$, and the projection on the xy plane looks like a triangle in the xy plane whose sides are the x-axis, the y-axis, and the line $2x + 3y = 12$ (the intersection of the plane $2x + 3y + 4z = 12$ with the xy plane $z = 0$). The next picture shows how we set up the x and y limits over this projection: $0 \le x \le 6$, and for fixed x we have $0 \le y \le \dfrac{12 - 2x}{3}$.



Therefore our integral becomes

$$\int_0^6 \int_0^{\frac{12-2x}{3}} \int_0^{\frac{12-2x-3y}{4}} f(x, y, z)\, dz\, dy\, dx.$$

Example: Rewrite the following integral in the order of integration $dx\, dy\, dz$, and then evaluate the integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2} z\, dz\, dy\, dx$$

First, we sketch the region. Start with the projection on the xy-plane. We have

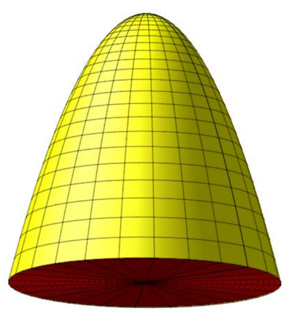
$$-1 \le x \le 1 \qquad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}.$$

This describes the unit disc.

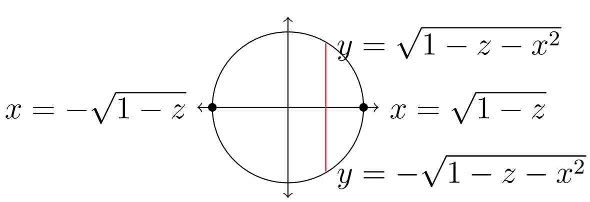
Now, for a fixed x and y, we have

$$0 \le z \le 1 - x^2 - y^2.$$

(Notice that for (x, y) in the unit disc, we do have $0 \le 1-x^2-y^2$, so this makes sense.) That is, our region is above the xy-plane and below the downward-facing paraboloid $z = 1-x^2-y^2$.



To write the integral in our new order, we can first look at the outer integral. At the top of the paraboloid $z = 1$, so $0 \le z \le 1$. A cross-section at a fixed z is a disc whose edge is on the paraboloid $z = 1 - x^2 - y^2$, which looks like a disc in the xy plane whose edge has equation $x^2 + y^2 = 1 - z$, which we can write $y = \pm\sqrt{1 - z - x^2}$, a circle of radius $\sqrt{1 - z}$.



This gives us

$$\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{-\sqrt{1-z-x^2}}^{\sqrt{1-z-x^2}} z\, dx\, dy\, dz.$$

To evaluate this integral, we can remember that the limits on x and y describe a disc around the origin of radius $\sqrt{1-z}$, which we will call R_z.

$$\int_0^1 \underbrace{\int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{-\sqrt{1-z-x^2}}^{\sqrt{1-z-x^2}} z\,dx\,dy}_{\iint_{R_z} z\,dA}\,dz.$$

$$\iint_{R_z} z\;dA = z\,(\mathrm{area(R_z)}) = z(\pi(1-z)) = \pi z - \pi z^2.$$

$$\int_0^1 \underbrace{\int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{-\sqrt{1-z-x^2}}^{\sqrt{1-z-x^2}} z\,dx\,dy}_{\iint_{R_z} z\,dA}\,dz = \int_0^1 (\pi z - \pi z^2)\,dz = \left(\frac{\pi z^2}{2} - \frac{\pi z^3}{3}\right)\Bigg|_{z=0}^{z=1} = \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{6}.$$

What does an integral like this mean?

Think of dividing the region D into tiny cubes of volume ΔV, assigning to each cube a number $f(x,y,z)\Delta V$ (where (x,y,z) is some point in that cube) and then adding up the results. In the limit, as the size of the cubes approaches 0, we get the integral. Whatever it is that the sum of the $f(x,y,z)\Delta V$ over all the tiny cubes approximates, this is what the integral represents.

Application 1: Suppose $f(x,y,z)=1$. Then we are just adding up the volumes of our tiny cubes, so

$$\iiint_D dV = \mathrm{volume(V)}.$$

Application 2: The average value of $f(x,y,z)$ over D is

$$\frac{1}{\mathrm{volume(D)}}\left(\iiint_D f(x,y,z)\,dV\right).$$

Application 3: Suppose that $f(x,y,z)$ represents the mass density at point (x,y,z), say in grams per cubic meter, of an object occupying region D. Then $f(x,y,z)\Delta V$ represents the approximate mass of a tiny cube of volume ΔV containing the point (x,y,z). Therefore, the mass of the object is given by the integral

$$\iiint_D f(x,y,z)\,dV.$$

The same applies to, for example, charge density.

The textbook gives several other physics applications. You don't need to memorize these formulas, but you should be able to use them, if they are given to you.

Example: The moment of inertia about an axis of a particle of mass m located a distance r from that axis is mr^2. If an object is composed of a large number of particles, the object's moment of inertia is the sum of the moments of inertia of the individual particles.

An object occupying the solid ball $x^2 + y^2 + z^2 \leq 1$ has mass density at point (x, y, z) of $f(x, y, z) = 2 - z^2$. Write down an integral representing its moment of inertia about the z-axis.

A tiny cube of volume ΔV containing point (x, y, z) will have moment of inertia about the z-axis approximately

$$(\text{distance from } z \text{ axis})^2(\text{mass}) = (\sqrt{x^2 + y^2}\,)^2(\underbrace{f(x, y, z)}_{\text{mass density}}\ \underbrace{\Delta V}_{\text{volume}}\,) = (x^2 + y^2)(2 - z^2)\Delta V.$$

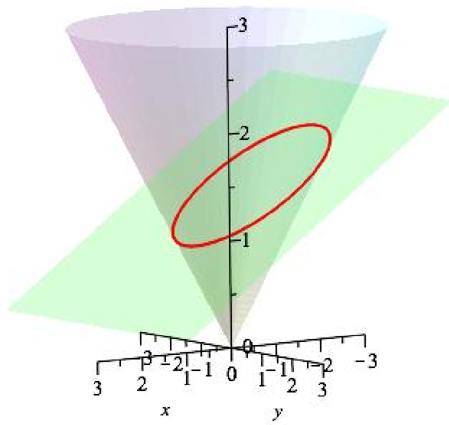
To approximate the object's moment of inertia we would add up the moments of inertia of these tiny cubes. We get the object's moment of inertia by taking a limit as the size of the cubes approaches 0. This is the triple integral

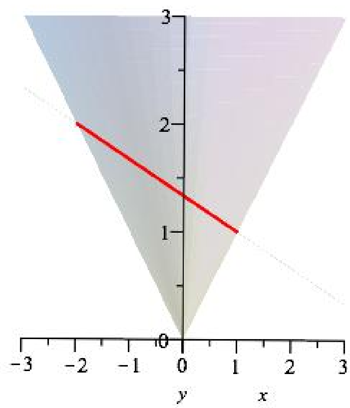
$$\iiint_{x^2+y^2+z^2\leq 1} (x^2 + y^2)(2 - z^2)\, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2)(2 - z^2)\, dz\, dy\, dx.$$

Next time we will learn how to use polar coordinates (or, more properly, spherical coordinates, which are three-dimensional coordinates (r, θ, z)) to compute an integral like this more easily than in rectangular coordinates.

Now for a challenge:

Example: Suppose D is the region above the cone $z = \sqrt{x^2+y^2}$ and below the plane $x + 3z = 4$. Express $\iiint_D f(x,y,z)\,dV$ as an iterated integral in two ways, with z as the inner variable of integration, and with z as the outer variable of integration.





Here are two different views of the cone and the plane, with their intersection drawn in red. The second view is straight along the y direction, so it shows a projection onto the xz plane. The region we want is the part of the cone below the plane; it is a solid cone with a slanted top surface.

With z as the inner variable of integration: The limits on z are the cone $z = \sqrt{x^2+y^2}$ below and the plane $z = \dfrac{4-x}{3}$ above, so we have

$$\iint_{\text{projection onto } xy \text{ plane}} \left[\int_{\sqrt{x^2+y^2}}^{\frac{4-x}{3}} f(x,y,z) dz \right] dA.$$

The projection onto the xy plane is bounded by the projection of the red ellipse onto the xy plane. To find that projection, we find points on both surfaces, where

$$z = \frac{4-x}{3} = \sqrt{x^2+y^2};$$

the projection is

$$\frac{4-x}{3} = \sqrt{x^2+y^2}.$$

By squaring both sides and then completing the square, we get

$$\frac{16-8x+x^2}{9} = x^2+y^2$$

$$8x^2+8x+9y^2 = 16$$

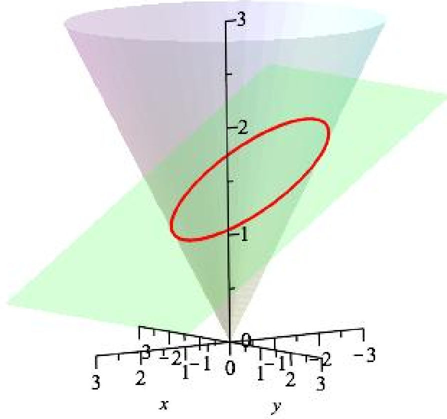
$$8\left(x+\frac{1}{2}\right)^2 + 9y^2 = 18.$$

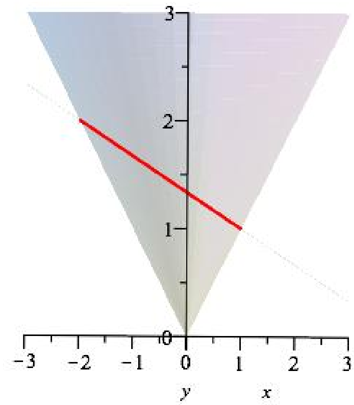
This is an ellipse[1] in the xy plane centered at $\left(-\frac{1}{2}, 0\right)$. The overall limits on x occur when $y=0$, at which point $x=-2$ or $x=1$. For fixed x, the limits on y are given by $8\left(x+\frac{1}{2}\right)^2 + 9y^2 = 18$, or $y = \pm\sqrt{2-\frac{8}{9}\left(x+\frac{1}{2}\right)^2}$. This gives us

$$\int_{-2}^{1} \int_{-\sqrt{2-\frac{8}{9}\left(x+\frac{1}{2}\right)^2}}^{\sqrt{2-\frac{8}{9}\left(x+\frac{1}{2}\right)^2}} \int_{\sqrt{x^2+y^2}}^{\frac{4-x}{3}} f(x,y,z)\, dz\, dy\, dx.$$

[1] An ellipse is one kind of conic section, which simply means that it is a shape you can get by slicing a cone with a plane. Other conic sections are parabolas and hyperbolas.

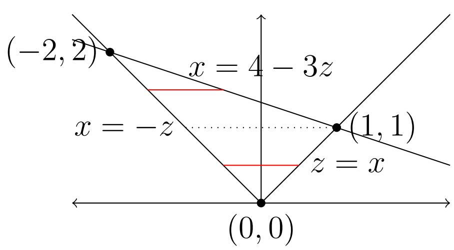
With z as the outer variable of integration: It will be convenient to use y as the inner variable of integration, and use the projection of our region onto the xz plane to find the limits on x and z.





It is clear from the pictures that, for fixed x and z, both the upper and lower limits on y are given by the cone $x^2 + y^2 = z^2$, or $y = \pm\sqrt{z^2 - x^2}$, so we have

$$\iint_{\text{projection onto } xz \text{ plane}} \left[\int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x,y,z)\, dy \right] dA.$$



From the picture, we see that the limits on x for fixed z are different when $0 \leq z \leq 1$ and when $1 \leq z \leq 2$, so we must write our integral as a sum:

$$\int_0^1 \int_{-z}^{z} \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x,y,z)\, dy\, dx\, dz + \int_1^2 \int_{-z}^{4-3z} \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x,y,z)\, dy\, dx\, dz.$$

Example: Write down an iterated (triple) integral representing the volume of the region above the surface $z = x^2 - 1$ and below the surface $z = 1 - y^2$.

Example: In the example on pages 8-10, we set up the integrals by thinking about the two-dimensional projections of D onto the xy and xz coordinate planes. Remember that another way to think about setting up integrals in three dimensions involves thinking about cross-sections of the region in question.

Draw typical cross-sections of D perpendicular to the x-axis (x is constant), perpendicular to the y-axis, perpendicular to the z-axis for $0 < z < 1$, and perpendicular to the z-axis for $1 < z < 2$. In each case, give equations for the boundary curves, and identify their points of intersection.

To get you started, here is a picture of one cross-section. You can figure out which one it is.