Math 11
Fall 2016
Section 1
Monday, October 17, 2016

First, some important points from the last class:

$$\iiint_D f(x,y,z)\,dV,$$

the integral (with respect to volume) of f over the three-dimensional region D, is a triple integral, defined as a limit of Riemann sums, and evaluated as an iterated integral:

$$\underbrace{\int_a^b}_{(x \text{ limits on entire region})} \underbrace{\int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)}}_{(y,z \text{ limits for fixed } x)} f(x,y,z)\,dz\,dy\,dx.$$

$$\underbrace{\int_a^b \int_{g_1(x)}^{g_2(x)}}_{(x,y \text{ limits on entire region})} \underbrace{\int_{h_1(x,y)}^{h_2(x,y)}}_{(z \text{ limits for fixed } x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Application 1: Suppose $f(x,y,z) = 1$. Then our Riemann sum is just adding up the volumes of our tiny cubes, so

$$\iiint_D dV = \text{volume(D)}.$$

Application 2: The average value of $f(x,y,z)$ over D is

$$\frac{1}{\text{volume(D)}}\left(\iiint_D f(x,y,z)\,dV\right).$$

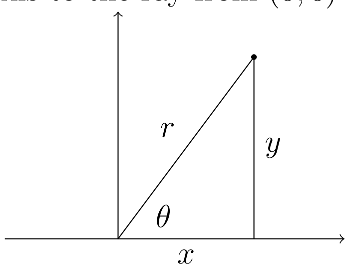
Application 3: Suppose that $f(x,y,z)$ represents the mass density at point (x,y,z), say in grams per cubic meter, of an object occupying region D. Then $f(x,y,z)\Delta V$ represents the approximate mass of a tiny cube of volume ΔV containing the point (x,y,z). Therefore, the mass of the object is given by the integral

$$\iiint_D f(x,y,z)\,dV.$$

The same applies to any density function.

Today: Integrals in polar and cylindrical coordinates.

Polar coordinates (r, θ): The distance from $(0,0)$ to (x,y) is r; the angle from the positive x-axis to the ray from $(0,0)$ to (x,y), measured counterclockwise, is θ.



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2+y^2}$$

Cylindrical coordinates (r, θ, z): Write x and y in polar coordinates.

Example: What curves do the following describe in polar coordinates:

$$r = 1 \qquad \theta = 0 \qquad \theta = \frac{\pi}{4} \qquad r = \theta \qquad r = \cos\theta \qquad ?$$

The unit circle; the positive x-axis; the ray from $(0,0)$ through $(1,1)$; a spiral; the line $x = 1$.

Example: What surfaces do the following describe in cylindrical coordinates:

$$r = 1 \qquad \theta = 0 \qquad r = z \qquad z = \theta \qquad r^2 + z^2 = 1 \qquad ?$$

The cylinder of radius 1 around the z-axis; the half-plane $y = 0$, $x \geq 0$; an upwards-facing cone with vertex at the origin; a spiral ramp; the unit sphere.

Example: To compute

$$\iint_D 1 - x^2 - y^2 \, dA,$$

where D is the unit disc:

This is the volume of the region above the xy-plane and below the paraboloid given by $z = 1 - x^2 - y^2$.

The region we are integrating over, the unit disc D, can be easily described in polar coordinates:

$$0 \leq \theta \leq 2\pi \qquad 0 \leq r \leq 1.$$

The function we are integrating can also be easily written in polar coordinates:

$$1 - x^2 - y^2 = 1 - r^2.$$

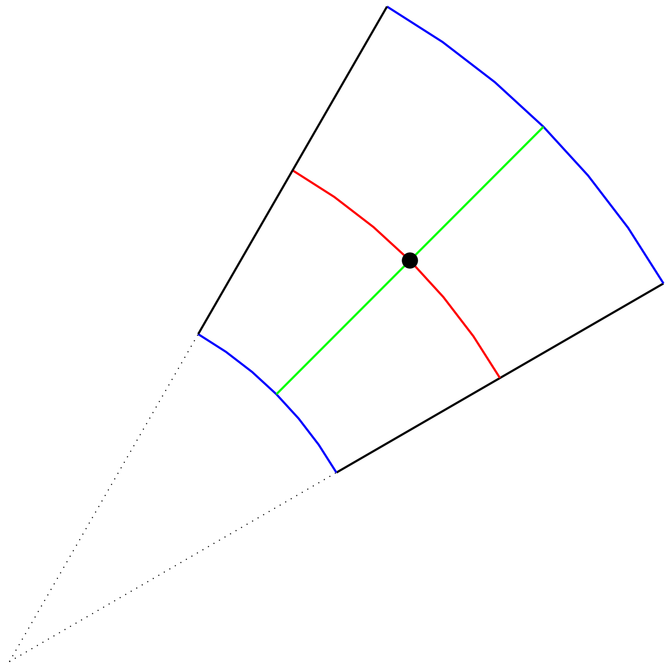
We can approximate this volume by using polar coordinates to divide D into small pieces: Break up the r interval into n subintervals of length Δr, and the θ subinterval into m subintervals of length $\Delta\theta$. This divides D into nm-many small regions. Let us say that region ij has area ΔA_{ij} and contains a point with polar coordinates (r_{ij}, θ_{ij}). Then we have

$$\iint_D 1 - x^2 - y^2 \, dA = \iint_D 1 - r^2 \, dA \approx \sum_{i=1}^{n} \sum_{j=}^{m} (1 - r^2) \, \Delta A_{ij};$$

$$\iint_D 1 - r^2 \, dA = \lim_{n,m \to \infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{m} (1 - r_{ij}^2) \, \Delta A_{ij} \right).$$

This looks almost like a limit of Riemann sums, but not quite, because ΔA_{ij} is not the same as $\Delta r \Delta\theta$.

We need to see what, in fact, ΔA_{ij} is.



This picture shows the small region labeled ij. Its boundaries are the blue and black edges. The marked point in the region has polar coordinates (r_{ij}, θ_{ij}). The dotted lines are emanating from the origin.

The difference in radius (distance from the origin) between the two curved blue edges is Δr. Therefore the green line has length Δr.

The angle between the straight black edges is $\Delta\theta$. Therefore, the length of the curved red arc is $r_{ij}\Delta\theta$. (The arc is the part of a circle of radius r_{ij} subtended by an angle $\Delta\theta$.)

Therefore the area of the small region shown is approximately the area of a rectangle of sides Δr and $r_{ij}\Delta\theta$, or $r_{ij}\Delta r\Delta\theta$. We use this approximation to rewrite our sum:

$$\iint_D 1 - r^2\, dA \approx \left(\sum_{i=1}^{n}\sum_{j=1}^{m}(1 - r_{ij}^2)\,\Delta A_{ij}\right) \approx \left(\sum_{i=1}^{n}\sum_{j=1}^{m}(1 - r_{ij}^2)\, r_{ij}\Delta r\Delta\theta\right).$$

Now this has the form of a Riemann sum:

$$\iint_D 1 - r^2\, dA = \lim_{n,m\to\infty}\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(1 - r_{ij}^2)\, r_{ij}\Delta r\Delta\theta\right) = \iint_D (1 - r^2) r\, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^1 (1 - r^2) r\, dr\, d\theta = \int_0^{2\pi}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=1} d\theta = \int_0^{2\pi}\frac{1}{4}\, d\theta = \frac{\pi}{2}.$$

The key thing to remember:

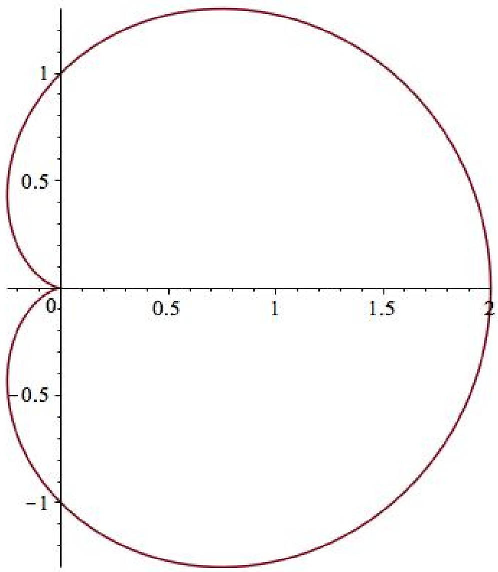
In rectangular coordinates, the differential area element is

$$dA = dx\, dy.$$

In polar coordinates, the differential area element is

$$dA = r\, dr\, d\theta.$$

Example: Find the area of the region inside the curve given in polar coordinates by $r = \cos\theta + 1$.



$$\int_0^{2\pi}\int_0^{1+\cos\theta} r\, dr\, d\theta = \int_0^{2\pi}\left(\frac{r^2}{2}\right)\Bigg|_{r=0}^{r=1+\cos\theta} d\theta = \int_0^{2\pi}\frac{1}{2} + \cos\theta + \frac{\cos^2\theta}{2}\, d\theta =$$

$$\int_0^{2\pi}\frac{1}{2}\, d\theta + \int_0^{2\pi}\cos\theta\, d\theta + \int_0^{2\pi}\frac{\cos^2\theta}{2}\, d\theta = \pi + 0 + \int_0^{2\pi}\frac{\cos(2\theta)+1}{4}\, d\theta = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

Cylindrical coordinates (r, θ, z):
TYPO IN CLASS NOTES – dz IS MISSING IN EXPRESSION FOR dV BELOW

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad dV = r\,dr\,d\theta\,dz.$$

Example (from last time):

The moment of inertia about an axis of a particle of mass m located a distance r from that axis is mr^2. If an object is composed of a large number of particles, the object's moment of inertia is the sum of the moments of inertia of the individual particles.

An object occupying the solid ball $x^2 + y^2 + z^2 \leq 1$ has mass density at point (x, y, z) of $f(x, y, z) = 2 - z^2$. Write down an integral representing its moment of inertia about the z-axis.

A tiny cube of volume ΔV containing point (x, y, z) will have moment of inertia about the z-axis approximately

$$(\text{distance from } z \text{ axis})^2(\text{mass}) = (\sqrt{x^2+y^2}\,)^2(\underbrace{f(x,y,z)}_{\text{mass density}}\ \underbrace{\Delta V}_{\text{volume}}) = (x^2+y^2)(2-z^2)\Delta V.$$

To approximate the object's moment of inertia we would add up the moments of inertia of these tiny cubes. We get the object's moment of inertia by taking a limit as the size of the cubes approaches 0. This is the triple integral

$$\iiint_{x^2+y^2+z^2\leq 1} (x^2+y^2)(2-z^2)\,dV.$$

$$\iiint_{x^2+y^2+z^2\le 1} (x^2+y^2)(2-z^2)\,dV.$$

Express the function to be integrated in cylindrical coordinates:

$$(x^2+y^2)(2-z^2) = r^2(2-z^2).$$

Express the region of integration in cylindrical coordinates (in two different ways):

$$0 \le \theta \le 2\pi \quad 0 \le r \le 1 \quad -\sqrt{1-r^2} \le z \le \sqrt{1-r^2};$$
$$-1 \le z \le 1 \quad 0 \le \theta \le 2\pi \quad 0 \le r \le \sqrt{1-z^2}.$$

Set up the integral (in two different ways):

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2(2-z^2)r\,dz\,dr\,d\theta;$$
$$\int_{-1}^1\int_0^{2\pi}\int_0^{\sqrt{1-z^2}} r^2(2-z^2)r\,dr\,d\theta\,dz;$$

Evaluate the integral: We'll evaluate the second one:

$$\int_{-1}^1\int_0^{2\pi}\int_0^{\sqrt{1-z^2}} r^2(2-z^2)r\,dr\,d\theta\,dz = \int_{-1}^1\int_0^{2\pi}\left((2-z^2)\frac{r^4}{4}\right)\Bigg|_{r=0}^{r=\sqrt{1-z^2}} d\theta\,dz =$$
$$\int_{-1}^1\int_0^{2\pi}\frac{(2-z^2)(1-z^2)^2}{4}\,d\theta\,dz = \int_{-1}^1 (2\pi)\frac{(-z^6+4z^4-5z^2+2)}{4}\,dz =$$
$$\frac{\pi}{2}\left(-\frac{z^7}{7}+\frac{4z^5}{5}-\frac{5z^3}{3}+2z\right)\Bigg|_{z=-1}^{z=1} = \frac{\pi}{2}\left(\left(-\frac{1}{7}+\frac{4}{5}-\frac{5}{3}+2\right)-\left(\frac{1}{7}-\frac{4}{5}+\frac{5}{3}-2\right)\right) = \frac{102\pi}{105}.$$

Example: Sketch the region of integration, then rewrite this integral in polar coordinates:

$$\int_0^{\frac{1}{2}} \int_{(\sqrt{3})y}^{\sqrt{1-y^2}} x\, dx\, dy.$$

$$\int_0^{\frac{\pi}{6}} \int_0^1 r^2 \cos\theta\, dr\, d\theta.$$

Example: Sketch the region of integration, then rewrite this integral in rectangular coordinates:

$$\int_0^{\frac{\pi}{3}} \int_0^{\frac{1}{\cos\theta}} r^2\, dr\, d\theta.$$

$$\int_0^1 \int_0^{\sqrt{3}\,x} \sqrt{x^2+y^2}\, dy\, dx$$

Example: Use an integral in cylindrical coordinates to find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.

Example: D is the region in the first octant inside the cylinder $x^2 + y^2 = 1$ and below the plane $z = 1$. Express

$$\iiint_D xy\, dV$$

in both cylindrical and rectangular coordinates. Then evaluate the integral, using whichever form you prefer.

Example: The following integral is given in cylindrical coordinates. Sketch the region of integration (as best you can) and rewrite the integral in rectangular coordinates. Then evaluate the integral, using whichever form you prefer.

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} \int_0^{1-r^2} r^2 \sin\theta \, dz \, dr \, d\theta$$