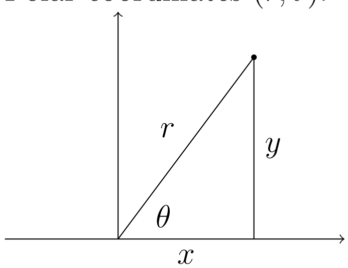
Math 11
Fall 2016
Section 1
Wednesday, October 19, 2016

First, some important points from the last class:

Polar coordinates (r, θ):



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

In rectangular coordinates, the differential area element is

$$dA = dx\,dy.$$

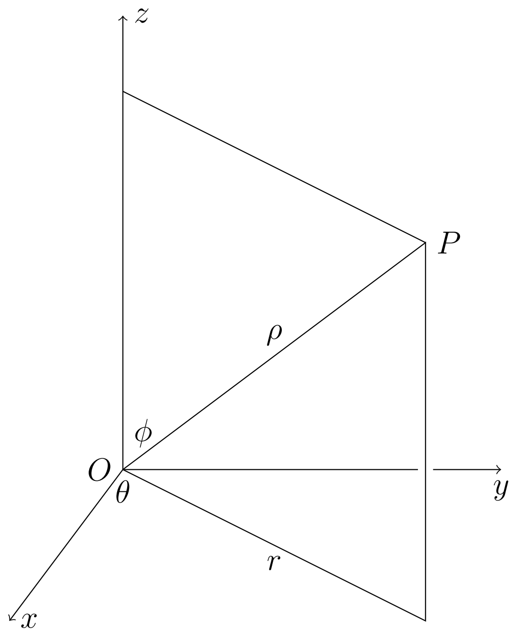
In polar coordinates, the differential area element is

$$dA = r\,dr\,d\theta.$$

Cylindrical coordinates (r, θ, z):

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad dV = r\,dr\,d\theta\,dz.$$

Today: Spherical coordinates (ρ, θ, ϕ):



$\rho =$ distance from origin to P

$\theta =$ polar coordinate of xy plane projection of P

$\phi =$ angle from positive z axis to $\overrightarrow{OP}$

$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$r = \rho \sin\phi$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

Example: What surfaces do the following describe in spherical coordinates:

$$\rho = 1 \qquad \theta = 0 \qquad \phi = \frac{\pi}{2} \qquad \phi = \frac{\pi}{4} \qquad \rho = \frac{1}{\sin\phi} \qquad \rho = 1 + \cos\phi \qquad \rho = \cos\phi \qquad ?$$

Unit sphere; half plane $y = 0, x \geq 0$; x plane; upward-facing cone; cylinder of radius 1 around z axis; surface obtained by revolving cardioid in xz plane around z-axis; sphere of radius $\frac{1}{2}$ and center $\left(0, 0, \frac{1}{2}\right)$.

Example: Find the volume of the three-dimensional region above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

In spherical coordinates, the sphere is $\rho = 1$, and the cone is $\phi = \frac{\pi}{4}$.

To be inside the sphere we need $0 \leq \rho \leq 1$, and to be above the cone we need $0 \leq \phi \leq \frac{\phi}{4}$.
Our limits on θ are $0 \leq \theta \leq 2\pi$.
The integral is

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{\sin\phi}{3} \, d\phi \, d\theta = \int_0^{2\pi} \left. \frac{-\cos\phi}{3} \right|_0^{\frac{\pi}{4}} d\theta = \frac{2\pi}{3} \left(-\frac{\sqrt{2}}{2} + 1 \right).$$

Example: Rewrite the following cylindrical coordinates integral in spherical coordinates:

$$\int_0^{2\pi}\int_1^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2\,dz\,dr\,d\theta.$$

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\frac{1}{\sin\phi}}^{2} (\rho\sin\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Example: Rewrite the following rectangular coordinates integral in spherical coordinates:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx.$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\phi\cos\theta+\sin\phi\sin\theta+\cos\phi}} (\rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Example: Rewrite the following spherical coordinates integral in cylindrical coordinates and in rectangular coordinates:

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\phi}} \rho^3 \sin^2\phi \, d\rho \, d\phi \, d\theta.$$

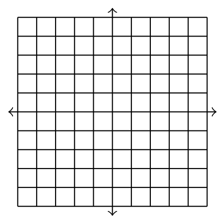
$$\int_0^{\frac{\pi}{4}} \int_0^{1} \int_r^{1} r^2 \, dz \, dr \, d\theta$$

$$\int_0^{\frac{\sqrt{2}}{2}} \int_y^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2} \, dz \, dx \, dy$$

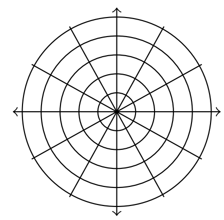
Example: Use an integral in spherical coordinates to find the volume of the region inside a spherical ball of radius a.

Example: An object occupying the unit ball has a mass density function $f(x, y, z) = z^2 + 1$. Find the object's total mass.

Exercise: We already have two different ways to assign coordinates to a point in the plane, rectangular coordinates and polar coordinates. In rectangular coordinates, dividing x- and y-intervals into subintervals of lengths Δx and Δy produces a grid in the plane, each rectangular patch having area $\Delta x\,\Delta y$. In polar coordinates, dividing r- and θ-intervals into subintervals of lengths Δr and $\Delta\theta$ produces a kind of grid in the plane (see the picture), each patch having area approximately $r\,\Delta r\,\Delta\theta$ (where (r,θ) are the polar coordinates of a point in the patch). We used this to write $dA = dx\,dy = r\,dr\,d\theta$.



$x = x \quad y = y \quad \Delta A = \Delta x\,\Delta y$



$x = r\cos\theta \quad y = r\sin\theta \quad \Delta A \approx r\,\Delta r\,\Delta\theta$

Consider another way of assigning coordinates, which we will call T coordinates (T for temporary; this is only for this problem). A point with the usual rectangular coordinates (x, y) has T coordinates (u, v) where $u = \dfrac{x}{2}$ and $v = \dfrac{y}{3}$.

If a point has T coordinates (u, v), what are its rectangular coordinates?

A rectangular region has corners with T coordinates $(u, v), (u + \Delta u, v)$, $(u, v + \Delta v)$, and $(u + \Delta u, v + \Delta v)$. What is the area ΔA of this region? (It is not $\Delta u\,\Delta v$. Try writing its corners in rectangular coordinates.)

To express a double integral in T coordinates, how should we express dA in terms of du and dv?

Describe the region in the xy plane whose area is given by $\displaystyle\iint_{(3x)^2+(2y)^2\le 36} dx\,dy$.

Rewrite this integral in T coordinates. (Use the same form; you need not write it as an iterated integral.)

Without actually computing an antiderivative, evaluate the integral.