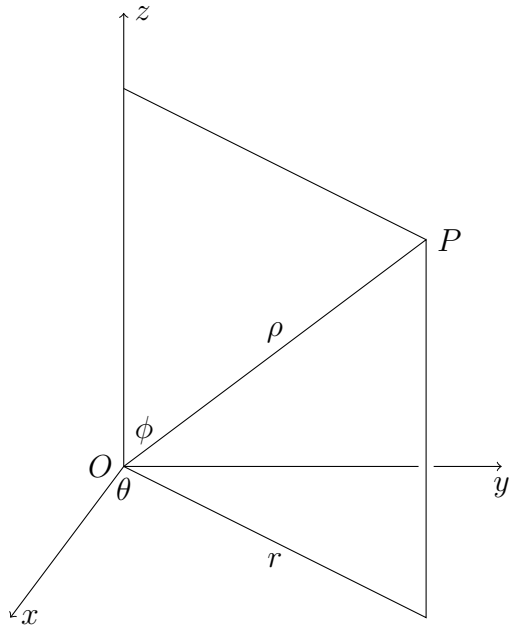


Math 11
Fall 2016
Section 1
Friday, October 21, 2016

First, some important points from the last class:

Spherical coordinates (ρ, θ, ϕ) :



ρ = distance from origin to P

θ = polar coordinate of xy plane projection of P

ϕ = angle from positive z axis to \overrightarrow{OP}

$$dV = \rho^2 \sin \phi d\rho d\theta d\phi$$

$$r = \rho \sin \phi$$

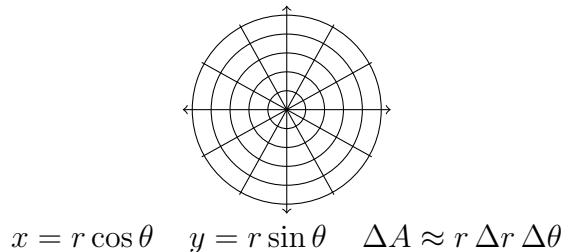
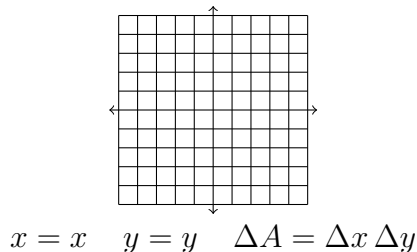
$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

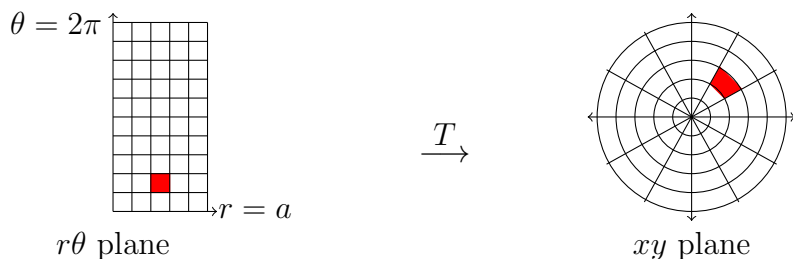
$$z = \rho \cos \phi$$

Today: General change of variables:

We already have two different ways to assign coordinates to a point in the plane, rectangular coordinates and polar coordinates. In rectangular coordinates, dividing x - and y -intervals into subintervals of lengths Δx and Δy produces a grid in the plane, each rectangular patch having area $\Delta x \Delta y$. In polar coordinates, dividing r - and θ -intervals into subintervals of lengths Δr and $\Delta \theta$ produces a kind of grid in the plane (see the picture), each patch having area approximately $r \Delta r \Delta \theta$ (where (r, θ) are the polar coordinates of a point in the patch). We used this to write $dA = dx dy = r dr d\theta$.



Another way to view this: We have a polar coordinate *transformation*, which is a function $T(r, \theta) = (x, y) = (r \cos \theta, r \sin \theta)$ that takes polar coordinates to rectangular coordinates. We can visualize T by drawing the $r\theta$ plane with a rectangular grid, and then drawing the xy plane as the image of the $r\theta$ plane with the transformed grid.



A small region of area $\Delta r \Delta \theta$ is transformed into a small region of area (approximately) $r \Delta r \Delta \theta$. Its area is stretched by a factor of r . The r in the expression $dA = r dr d\theta$ is precisely this factor by which areas are stretched by the transformation T .

For a general change of variable, to evaluate an integral $\iint_R f(x, y) dA$, we will relate rectangular coordinates (x, y) to some other coordinates (u, v) by means of a transformation T , setting $(x, y) = T(u, v)$. By writing (x, y) in terms of (u, v) , we can rewrite both the region of integration R and the integrand $f(x, y)$ in terms of u and v . (We choose T so this rewriting simplifies things.) To rewrite dA in terms of du and dv , we need a technique to find the stretching factor for the transformation T .

Note that in general this stretching factor will not be a number, but will be a function of (u, v) . For the polar coordinate transformation, we have $\text{stretch}(r, \theta) = r$.

Later we will learn the official name of the stretch function.

For example, let's look at another way of assigning coordinates, which we will call T coordinates. (T for temporary; this is only for this problem.) A point with the usual rectangular coordinates (x, y) has T coordinates (u, v) where $u = \frac{x}{2}$ and $v = \frac{y}{3}$.

If a point has T coordinates (u, v) , what are its rectangular coordinates?

$$(x, y) = (2u, 3v)$$

A rectangular region has corners with T coordinates (u, v) , $(u + \Delta u, v)$, $(u, v + \Delta v)$, and $(u + \Delta u, v + \Delta v)$. What is the area ΔA of this region? (It is not $\Delta u \Delta v$. Try writing its corners in rectangular coordinates.)

Corners $(2u, 3v)$, $(2u + 2\Delta u, 3v)$, $(2u, 3v + 3\Delta v)$, and $(2u + 2\Delta u, 3v + 3\Delta v)$; sides $2\Delta u$ and $3\Delta v$; area $\Delta A = 6\Delta u \Delta v$.

To express a double integral in T coordinates, how should we express dA in terms of du and dv ?

$$dA = 6 du dv.$$

Describe the region in the xy plane whose area is given by $\iint_{(3x)^2 + (2y)^2 \leq 36} dx dy$.

$(3x)^2 + (2y)^2 \leq 36$ gives the region inside an ellipse through points $(\pm 2, 0)$ and $(0, \pm 3)$.

Rewrite this integral in T coordinates. (Use the same form; you need not write it as an iterated integral.)

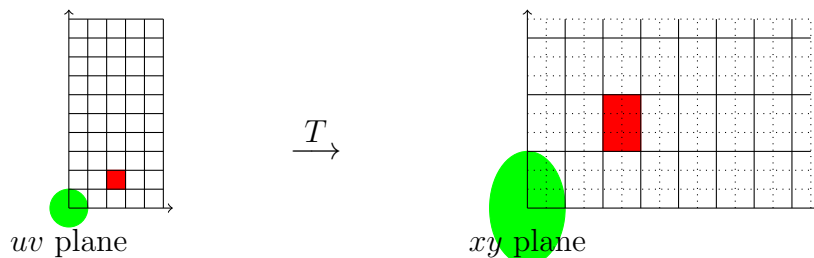
The region of integration is $(3(2u))^2 + (2(3v))^2 \leq 36$, or $u^2 + v^2 \leq 1$. $\iint_{u^2 + v^2 \leq 1} 6 du dv$

Without actually computing an antiderivative, evaluate the integral.

$$6(\text{area of region of integration}) = 6\pi.$$

This change of variables is also a transformation. (Okay, that's what T really stands for.) We have $(x, y) = T(u, v) = (2u, 3v)$, and the stretching factor is 6.

The stretching factor here makes perfect sense; the u axis is stretched out by a factor of 2, and the v axis by a factor of 3. Stretching the unit circle in this way gives an ellipse.



Our strategy from this point:

First, use what we know about vectors and geometry to find the stretching factor for a linear transformation $T(u, v) = (au + bv + p, cu + dv + q)$.

Second, say that any transformation T (as long as T is a differentiable function) is approximated near a point by its linear approximation, and in the limit, the stretching factor for T is the same as the stretching factor for the linear approximation.

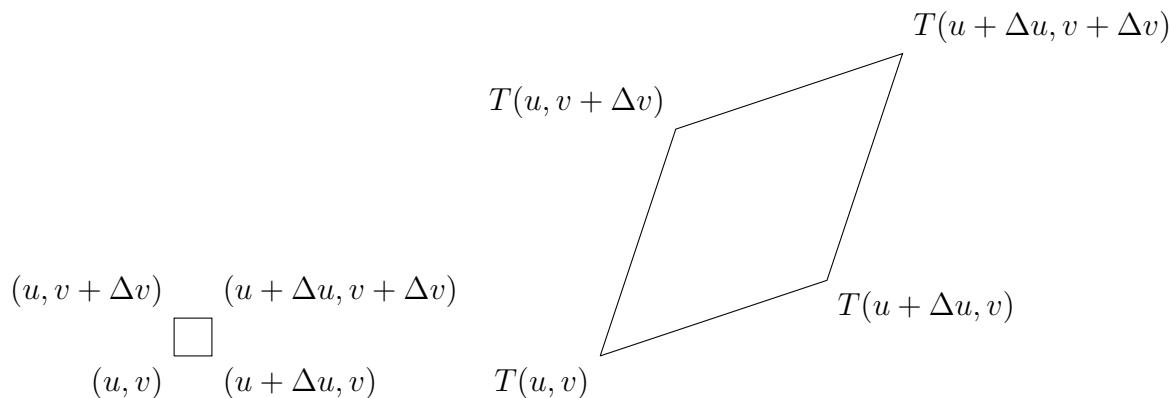
Third, note that the same ideas work in three dimensions. (Also in more than three dimensions, if we accept that the stretching factor for linear transformations works the same way.)

Spoiler: The stretching factor is the absolute value of the determinant of a matrix of partial derivatives.

First, consider a linear transformation given by

$$(x, y) = (x(u, v), y(u, v)) = T(u, v) = (au + bv + p, cu + dv + q).$$

Pictured here are a small rectangle in the uv plane of dimensions $\Delta u \times \Delta v$, and its transformed version (its image) in the xy -plane. We want to know the area of the image.



The image is a parallelogram. One edge is the vector from $T(u, v)$ to $T(u + \Delta u, v)$. We find the coordinates of the vector by subtracting the corresponding coordinates of the points:

$$\langle a(u + \Delta u) + bv + p, c(u + \Delta u), dv, q \rangle - \langle a(u) + bv + p, c(u), dv, q \rangle = \langle a\Delta u, c\Delta u \rangle.$$

Similarly, the adjacent edge of the parallelogram is the vector from $T(u, v)$ to $T(u, v + \Delta v)$, which is $\langle b\Delta v, d\Delta v \rangle$.

We need to know the area of the parallelogram with edges $\langle a\Delta u, c\Delta u \rangle$ and $\langle b\Delta v, d\Delta v \rangle$.

We need to know the area of the parallelogram with edges $\langle a\Delta u, c\Delta u \rangle$ and $\langle b\Delta v, d\Delta v \rangle$.

We can find this by adding a z -coordinate of zero (to view the xy plane as sitting inside \mathbb{R}^3), and using the cross product to find the area of the parallelogram:

$$|\langle a\Delta u, c\Delta u, 0 \rangle \times \langle b\Delta v, d\Delta v, 0 \rangle| = |\Delta u(\langle a, c, 0 \rangle) \times \Delta v(\langle b, d, 0 \rangle)| = |\langle a, c, 0 \rangle \times \langle b, d, 0 \rangle| \Delta u \Delta v =$$

$$\left| \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & c & 0 \\ b & d & 0 \end{pmatrix} \right| \Delta u \Delta v = |\langle 0, 0, ad - bc \rangle| \Delta u \Delta v = |ad - bc| \Delta u \Delta v = \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| \Delta u \Delta v.$$

We can identify a , b , c , and d as partial derivatives of the coordinate functions of the transformation T :

$$(x, y) = T(u, v) = (au + bv + p, cu + dv + q);$$

$$a = \frac{\partial x}{\partial u} \quad b = \frac{\partial x}{\partial v} \quad c = \frac{\partial y}{\partial u} \quad d = \frac{\partial y}{\partial v}.$$

If T were any other differentiable transformation with the same partial derivatives (we are guaranteed that T is differentiable if all these partial derivatives are continuous), in the limit we would have the same stretching factor:

$$\Delta A \approx \left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| \Delta u \Delta v = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| \Delta u \Delta v.$$

The determinant of the matrix of partial derivatives has a special name:

Definition: The *Jacobian* of a transformation $(x, y) = T(u, v)$ is the determinant of the matrix of partial derivatives. It is denoted

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Proposition: If $(x, y) = T(u, v)$ is a differentiable transformation, then when changing variables in an integral from (x, y) to (u, v) , we have

$$dx dy = dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv.$$

That is, the stretching factor is the absolute value of the Jacobian.

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \text{(row for } x\text{)} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \text{(row for } y\text{)} \\ \text{(column for } \frac{\partial}{\partial u}\text{)} & \text{(column for } \frac{\partial}{\partial v}\text{)} & \end{pmatrix}.$$

Example: In polar coordinates $(x, y) = (r \cos \theta, r \sin \theta)$. Then

$$\begin{aligned} dA &= \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta = \left| \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right| dr d\theta = \left| \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \right| dr d\theta = \\ &= |r \cos^2 \theta + r \sin^2 \theta| dr d\theta = r dr d\theta. \end{aligned}$$

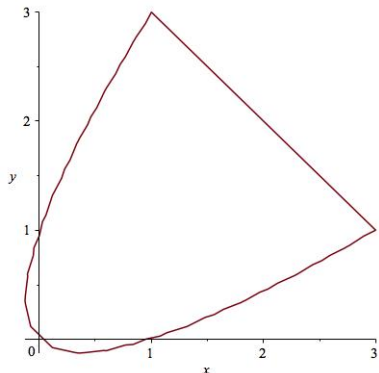
Example: In the previous transformation $(x, y) = (2u, 3v)$ that transformed the unit disc $u^2 + v^2 \leq 1$ into the elliptical region $(3x)^2 + (2y)^2 \leq 36$, we have

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| du dv = \left| \det \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right| du dv = 6 du dv.$$

Note: In three dimensions, we have the analogous stretching factor.

$$dA = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \right| du dv dw.$$

Example: Find the area of the region R bounded by the curve $x + y = (x - y)^2$ and the line $x + y = 4$.



Use a change of variable

$$u = x - y \quad v = x + y.$$

Then our boundary curves become

$$v = u^2 \quad v = 4$$

and the area is given by the integral

$$\iint_R dA = \int_{-2}^2 \int_{u^2}^4 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dv du.$$

To find the Jacobian, we need to solve for x and y in terms of u and v :

$$x = \frac{u + v}{2} \quad y = \frac{v - u}{2} \quad \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}.$$

Now we have

$$\int_{-2}^2 \int_{u^2}^4 \left(\frac{1}{2} \right) dv du = \int_{-2}^2 \left(2 - \frac{u^2}{2} \right) du = \left(2u - \frac{u^3}{6} \right) \Big|_{u=-2}^{u=2} = \frac{16}{3}.$$

Example: Find the volume of the region above the region R inside the parallelogram with vertices (in order counterclockwise) $(0, 0)$, $(3, 1)$, $(5, 4)$, $(2, 3)$, and below the surface $z = x + y$.

Compute

$$\iint_R (x + y) dA$$

by using an appropriate change of variable.

There are (at least) two ways to think about finding the change of variable:

I. We already saw that a linear change of variable $(x, y) = (au + bv + p, cu + dv + q)$ transforms the unit square to a parallelogram. To send $(u, v) = (0, 0)$ to $(x, y) = (0, 0)$, we need $p = 0$ and $q = 0$. To then send $(u, v) = (1, 0)$ to $(x, y) = (3, 1)$ we need $a = 3$ and $c = 1$. To send $(u, v) = (0, 1)$ to $(x, y) = (2, 3)$ we need $b = 2$ and $d = 3$. Therefore, our transformation is

$$x = 3u + 2v \quad y = u + 3v$$

our Jacobian is

$$\det \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix} = 7$$

and our integral is

$$\iint_R (x + y) dA = \int_0^1 \int_0^1 (4u + 5v) 7 du dv.$$

II. The four lines that are the boundaries of our parallelogram have equations

$$3y - x = 0 \quad 3y - x = 7 \quad 2y - 3x = 0 \quad 2y - 3x = -7.$$

Use a transformation

$$u = 2y - 3x \quad v = 3y - x \quad x = \frac{-3u + 2v}{7} \quad y = \frac{3v - u}{7}.$$

Our Jacobian is

$$\det \begin{pmatrix} \frac{-3}{7} & \frac{2}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{pmatrix} = \frac{-1}{7}$$

and our integral is

$$\iint_R (x + y) dA = \int_0^7 \int_{-7}^0 \left(\frac{5v - 4u}{7} \right) \frac{1}{7} du dv.$$

Example: Show that the Jacobian of the spherical coordinate transformation

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

is

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin \phi.$$

Example: Sometimes the function being integrated calls for a change of variable.
Evaluate

$$\int_0^4 \int_0^4 x \sin(\sqrt{y}) dy dx$$

by using a change of variable

$$u = x \quad v = \sqrt{y}$$

Example: Use a suitable change of variable to find the volume of the region above the xy -plane and below the surface

$$z = 4 - (x + y)^2 - (2x - y)^2.$$