Math 11
Fall 2016
Section 1
Monday, October 24, 2016

First, some important points from the last class:

Definition: The *Jacobian* of a transformation $(x, y) = T(u, v)$ is the determinant of the matrix of partial derivatives. It is denoted

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

Proposition: If $(x, y) = T(u, v)$ is a differentiable transformation, then when changing variables in an integral from (x, y) to (u, v), we have

$$dx\,dy = dA = \left| \frac{\partial(x,y)}{\partial(u,v))} \right| du\,dv.$$

That is, the stretching factor is the absolute value of the Jacobian.

In three dimensions, we have the analogous stretching factor.

$$dA = \left| \frac{\partial(x,y,z)}{\partial(u,v,w))} \right| du\,dv\,dw = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \right| du\,dv\,dw.$$

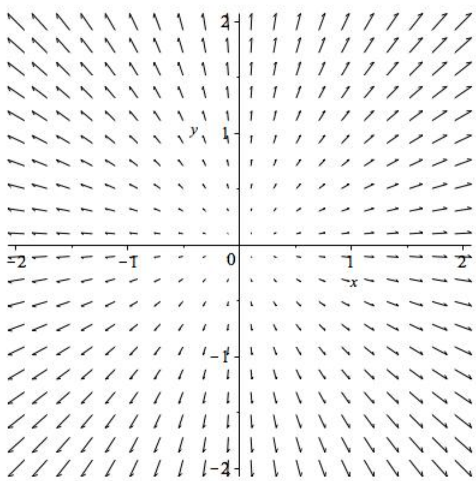
Today: We revisit two things we have already seen, gradient fields and arc length integrals, and generalize them.

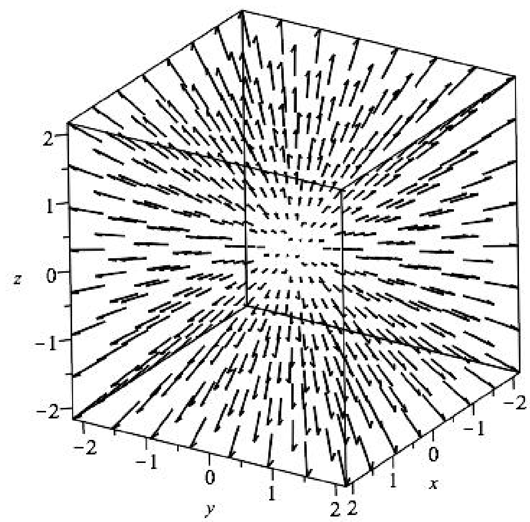
Recall that if $f : \mathbb{R}^2 \to \mathbb{R}$, then $\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$, and we picture ∇f as assigning to each point (x, y) in $\mathbb{R}^2$ a vector $\langle f_x(x, y), f_y(x, y)\rangle$ in $\mathbb{R}^2$.

We can draw a picture of ∇f by drawing the vector $\nabla f(x, y)$ emanating from the point (x, y), for a number of points. Often, as in most of these pictures, we scale the vectors so they don't overwhelm the picture.

We have similar definitions and pictures in $\mathbb{R}^3$, and similar definitions, but no pictures, in $\mathbb{R}^n$ for larger n.

Here are pictures of ∇f for $f(x, y) = x^2 + y^2$ and for $f(x, y, z) = x^2 + y^2 + z^2$.





Definition: An n-dimensional *vector field* is a function $F : \mathbb{R}^n \to \mathbb{R}^n$. We think of it as assigning to each point in n-dimensional space a vector in n-dimensional space.

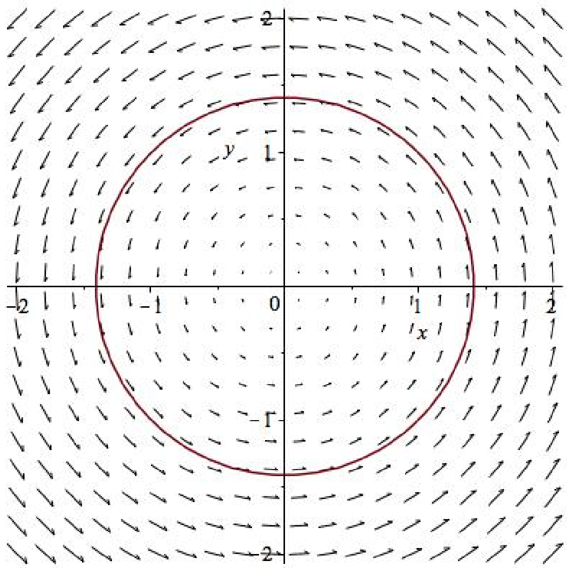
Other examples of vector fields include fluid flow fields ($F(\vec{r})$ is the velocity of flow at the point $\vec{r}$ — meaning, of course, at the point with position vector $\vec{r}$), force fields ($F(\vec{r})$ is the force exerted at point $\vec{r}$), gravitational fields ($F(\vec{r})$ is the gravitational force exerted on a unit mass located at point $\vec{r}$), and electric fields ($F(\vec{r})$ is the electric force exerted on a unit charge located at point $\vec{r}$).

Every gradient field is a vector field, but not all vector fields are gradient fields.

Definition: A vector field F is called *conservative* if it is a gradient field. If we write $F = -\nabla f$, then f is a *potential function* for f.

We will see in a bit exactly what this terminology means. If you have studied physics, you may already have an idea.

Example: Pictured are a vector field and a circle around the origin.



This is not a conservative vector field, and here is one way we can tell:

Suppose this were the gradient field of f. Imagine walking on the graph of f, on a path above the pictured circle. As you walk, you are going in the direction of the gradient, so you are walking uphill on the surface. This is true all along the circle. After going around the circle once, you are back where you started, but were walking uphill all the time.

Unless you inhabit a print by M.C. Escher or something, this is impossible.

Suppose F is a vector field in $\mathbb{R}^2$. If $F = \nabla f$, then a closed curve (a curve whose beginning point equals its end point) γ in the xy-plane is the projection of a closed curve on the graph of f. When you go around that curve, the net change in altitude is zero.

Thinking of γ as your horizontal path, if you integrate the slope of your path on the graph of f around γ, you should get the net change in altitude, which should be zero. The slope of your path is the directional derivative in the direction in which you are going, or $D_{\vec{T}}f$, where $\vec{T}$ is the unit tangent vector to γ. We know how to compute the directional derivative: $D_{\vec{T}}f = \nabla f \cdot \vec{T}$. If $F = \nabla f$, this is $F \cdot \vec{T}$. So we want to say:

If F is a conservative vector field in $\mathbb{R}^2$, and γ is a closed curve in $\mathbb{R}^2$, then the integral of $F \cdot \vec{T}$ around γ must equal zero.

Before we can say anything like this, we must have some idea of what it means to integrate a function along a curve.

In the picture above, γ is a circle of radius $\sqrt{2}$, and at every point on γ, the vector $F(x, y)$ is tangent to γ and has length $\sqrt{2}$. Therefore, we have

$$F \cdot \vec{T} = |F|\,|\vec{T}|\,cos(\theta) = \sqrt{2}.$$

Integrating the constant $\sqrt{2}$ along a curve γ of length $2\pi\sqrt{2}$ ought to give the constant times the length of the curve, or $(\sqrt{2})(2\pi\sqrt{2}) = 4\pi$.

In particular, walking a horizontal distance of $2\pi\sqrt{2}$ on a path of slope $\sqrt{2}$ produces an altitude gain of $(\sqrt{2})(2\pi\sqrt{2}) = 4\pi$.

The integral of a constant function with value c along a curve γ should be c times the length of γ.

Suppose that f is not constant, and γ is parametrized by a function $\vec{r}(t)$ for $a \leq t \leq b$. We approximate the integral of f along γ by dividing $[a, b]$ into n subintervals of length Δt, which has the effect of chopping γ up into n-many tiny curves.

We choose a point t_i^* in the i^{th} subinterval. Then $\vec{r}_i^{\,*} = \vec{r}(t_i^*)$ is a point on the i^{th} tiny piece of γ, and on that piece, the value of f is approximately $f(\vec{r}_i^{\,*}) = f(\vec{r}(t_i^*))$. If that piece of γ has length Δs_i, then the integral of f along that piece is approximately $f(\vec{r}_i^{\,*})\,\Delta s_i = f(\vec{r}(t_i^*))\,\Delta s_i$.

The length of the i^{th} piece of γ is $\Delta s_i \approx |\vec{r}\,'(t_i^*)|\,\Delta t$. (If we think of $\vec{r}$ as a position function, this is speed at time t_i^* times elapsed time to travel over the i^{th} piece of γ.) Therefore, the integral of f over the i^{th} piece of γ is approximately $f(\vec{r}_i^{\,*})\Delta s_i \approx f(\vec{r}(t_i^*))\,|\vec{r}\,'(t_i^*)|\,\Delta t$.

The integral of f over γ is the sum of the integrals of f over the tiny pieces, or approximately

$$\sum_{i=1}^{n} f(\vec{r}_i^{\,*})\Delta s_i \approx \sum_{i=1}^{n} f(\vec{r}(t_i^*))\,|\vec{r}\,'(t_i^*)|\,\Delta t.$$

In the limit, we get the actual value:

$$\int_\gamma f\,ds = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(\vec{r}_i^{\,*})\Delta s_i\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(\vec{r}(t_i^*))\,|\vec{r}\,'(t_i^*)|\,\Delta t\right) = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt.$$

In the integral on the left, ds means that we are integrating with respect to arc length. The integral on the right lets us evaluate this arc length integral by using a parametrization of γ to transform it into an ordinary integral. We can think

$$ds = |\vec{r}\,'(t)|\,dt.$$

Applications:

$$\int_\gamma 1\,ds = \text{ arc length of } \gamma.$$

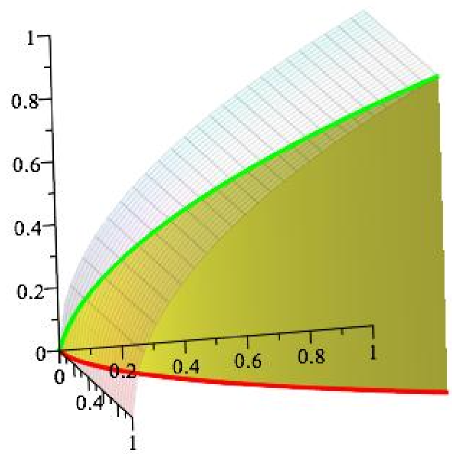
$$\frac{1}{\text{arc length of } \gamma}\int_\gamma f\,ds = \text{ average value of } f \text{ on } \gamma.$$

$$\int_\gamma \text{linear mass density (grams per meter) } ds = \text{ mass of } \gamma.$$

If γ is a curve in $\mathbb{R}^2$ and f is non-negative, then

$$\int_\gamma f\,ds = \text{ area of fence with base } \gamma \text{ and height given by } f.$$

Example: Find the area of the surface given by $y = x^2$, $0 \le x \le 1$, $0 \le z \le \sqrt{y}$.



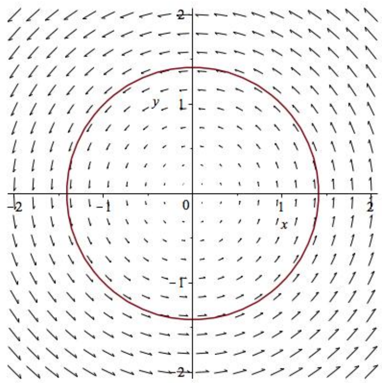
If γ is the portion of the parabola $y = x^2$ for $0 \le x \le 1$, then this is a vertical fence with base γ, and top edge on the surface $z = \sqrt{y}$.

Parametrize γ by $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$, $\vec{r}\,'(t) = \langle 1, 2t \rangle$, $|\vec{r}\,'(t)| = \sqrt{1 + 4t^2}$.

The area is

$$\int_\gamma \sqrt{y}\, ds = \int_0^1 \sqrt{t^2}\,\sqrt{1+4t^2}\, dt = \int_0^1 t(1+4t^2)^{\frac{1}{2}}\, dt = \left(\frac{1}{12}(1+4t^2)^{\frac{3}{2}}\right)\Bigg|_{t=0}^{t=1} = \frac{5^{\frac{3}{2}} - 1}{12}.$$

Example: Find $\int_\gamma F \cdot \vec{T}\, ds$, where γ is the circle $x^2 + y^2 = 2$ oriented counterclockwise, $\vec{T}$ is the unit tangent vector to γ, and $F(x,y) = \langle -y, x \rangle$. (Notice, we can tell this is not a gradient field, by Clairaut's Theorem.)



Note: In the applications listed on page 4, the orientation of γ does not matter. In this case it does, because if we change the orientation of γ, we reverse the direction of $\vec{T}$, so we change the sign of the function $F \cdot \vec{T}$ that we are integrating.

Parametrize γ, which is a circle of radius $\sqrt{2}$:

$$\vec{r}(t) = \left\langle \sqrt{2}\cos(t), \sqrt{2}\sin(t) \right\rangle \qquad 0 \le t \le 2\pi;$$

$$\vec{r}\,'(t) = \left\langle -\sqrt{2}\sin(t), \sqrt{2}\cos(t) \right\rangle \qquad |\vec{r}\,'(t)| = \sqrt{2} \qquad \vec{T} = \langle -\sin(t), \cos(t) \rangle \qquad ds = \sqrt{2}\, dt;$$

$$\int_\gamma F \cdot \vec{T}\, ds = \int_\gamma \langle -y, x \rangle \cdot \vec{T}\, ds = \int_0^{2\pi} \left\langle -\sqrt{2}\sin(t), \sqrt{2}\cos(t) \right\rangle \cdot \langle -\sin(t), \cos(t) \rangle \sqrt{2}\, dt =$$

$$\int_0^{2\pi} \sqrt{2}\sqrt{2}\, dt \int_0^{2\pi} 2\, dt = 4\pi.$$

Note: The function we are integrating, $F \cdot \vec{T}$, is defined only for points (x,y) on γ, since $\vec{T}$, the unit tangent vector to γ, is defined only for points on γ. This is okay.

Doing this in general leads to another kind of integral along γ:

Suppose we want to compute

$$\int_\gamma F \cdot \vec{T}\, ds,$$

where $F(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a vector function on $\mathbb{R}^2$, and γ is a smooth curve in $\mathbb{R}^2$ parametrized by $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$.

On γ, we write

$$\begin{aligned} F &= \langle P(x(t),y(t)), Q(x(t),y(t))\rangle \\ \vec{T} &= \frac{1}{|\langle x'(t), y'(t)\rangle|}\langle x'(t), y'(t)\rangle \\ ds &= |\langle x'(t), y'(t)\rangle|\, dt \end{aligned}$$

$$\int_\gamma F\cdot\vec{T}\, ds = \int_a^b \langle P(x(t),y(t)), Q(x(t),y(t))\rangle\cdot\left(\frac{1}{|\langle x'(t), y'(t)\rangle|}\langle x'(t), y'(t)\rangle\right)|\langle x'(t), y'(t)\rangle|\, dt =$$

$$\int_a^b \left(P(x(t),y(t))\, x'(t) + Q(x(t),y(t))\, y'(t)\right)\, dt =$$

$$\int_a^b \left(P(x(t),y(t))\, \underbrace{x'(t)\, dt}_{dx} + Q(x(t),y(t))\, \underbrace{y'(t)\, dt}_{dy}\right) =$$

$$\int_\gamma P\, dx + Q\, dy = \int_\gamma P\, dx + \int_\gamma Q\, dy,$$

where

$$\int_\gamma P\, dx = \int_a^b P(x(t),y(t))\, x'(t)\, dt \qquad \int_\gamma Q\, dy = \int_a^b Q(x(t),y(t))\, y'(t)\, dt.$$

We sometimes write

$$\vec{T}\, ds = \langle dx, dy\rangle \qquad \int_\gamma F\cdot\vec{T}\, ds = \int_\gamma F\cdot\langle dx, dy\rangle\, .$$

Note: In $\int_\gamma P\, dx$ and $\int_\gamma Q\, dy$, the curve γ must be an *oriented* curve. Direction matters.

Example: Compute $\int_\gamma -y\,dx$, where γ is the curve $x^2+y^2=2$, oriented counterclockwise.

This is part of the integral

$$\int_\gamma \langle -y, x\rangle \cdot \vec{T}\,ds = \int_\gamma -y\,dx + x\,dy$$

we computed earlier.

Parametrize γ:

$$\langle x, y\rangle = \left\langle \sqrt{2}\cos(t), \sqrt{2}\sin(t)\right\rangle \qquad 0 \le t \le 2\pi \qquad dx = -\sqrt{2}\sin(t)\,dt$$

$$\int_\gamma -y\,dx = \int_0^{2\pi} \underbrace{(-\sqrt{2}\sin(t))}_{-y}\,\underbrace{(-\sqrt{2}\sin(t))\,dt}_{dx} = \int_0^{2\pi} 2\sin^2(t)\,dt = (1-\cos(2t))\Big|_{t=0}^{t=2\pi} = 2\pi.$$

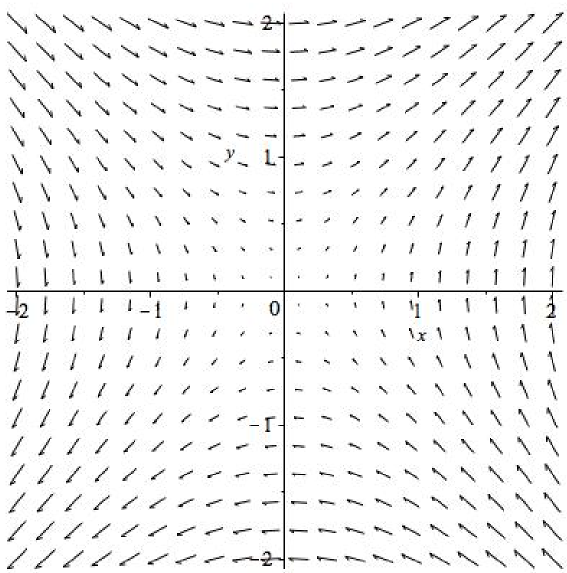
Example: Find the average y-coordinate of a point on γ, the top half of the unit circle.

$$\left(\frac{1}{\text{arc length of } \gamma}\right) \int_\gamma y\, ds.$$

Parametrize γ by $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$; $0 \leq t \leq \pi$; $\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$; $|\vec{r}\,'(t)| = 1$.

$$\frac{1}{\pi} \int_\gamma y\, ds = \frac{1}{\pi} \int_0^\pi (\sin(t))\,(1)\, dt = -\frac{1}{\pi} \cos(t)\Big|_{t=0}^{t=\pi} = \frac{2}{\pi} \approx .637.$$

Example: Sketch the vector field $F(x, y) = \left\langle \frac{y}{10}, \frac{x}{10} \right\rangle$.



Compute $\int_\gamma y\,dx + x\,dy$, where γ is the portion of the graph $y = \cos(x)$ for $0 \le x \le \frac{\pi}{4}$.

Example: The function f is defined for $x < 0$ by

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right).$$

Compute $\nabla f(x,y)$. Sketch the gradient field of f.

(To picture what's going on, try rewriting f using polar coordinates.)

Define a vector field $F(x,y)$ by using the formula you found above for $\nabla f(x,y)$. What is the domain of F?

Compute $\displaystyle\int_\gamma F \cdot \vec{T}\, ds$, where γ is the unit circle oriented counterclockwise. Your answer should not be zero. This is because, even though F can be represented as a gradient field on part of $\mathbb{R}^2$, it cannot be represented as a gradient field on any open region that includes the entire unit disc.