Math 11

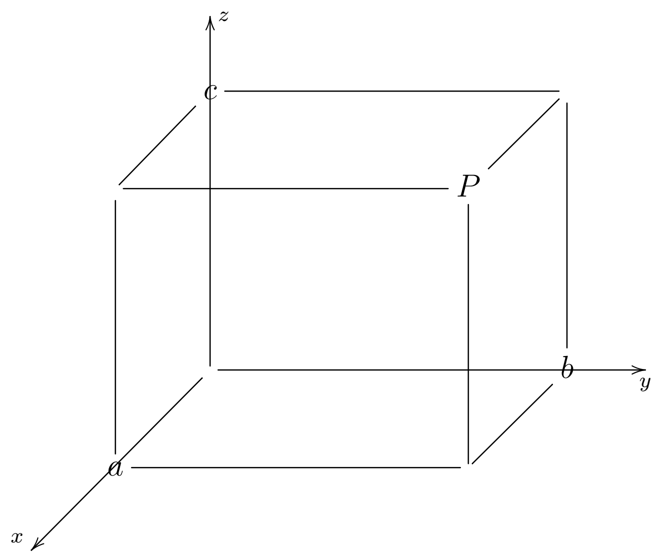
Fall 2016

Section 1

Wednesday, September 14, 2016

First, some important points from the last class:

Three-dimensional coordinate system (x-axis points out of paper):



$$P = (a, b, c)$$

Distance from $Q = (x_1,\, y_1,\, z_1)$ to $P = (x_2,\, y_2,\, z_2)$ is

$$\boxed{|QP| = \sqrt{(x^2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

$\mathbb{R}^3$ denotes 3-dimensional space, or the set of all triples (a, b, c) of real numbers.

Sphere of radius r with center (a, b, c):

$$\boxed{(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2}$$

Vectors are used to model anything that has magnitude (represented by the length of the vector) and direction (represented by the direction of the vector).

Example: The displacement of an object moving from point $P = (a_1, b_1, c_1)$ to point $Q = (a_2, b_2, c_2)$ is represented by the displacement vector $\langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle$.

Vectors, vector addition, scalar multiplication, subtraction, magnitude (norm), and direction have algebraic and geometric representations, and different meanings in different applications.

Two arrows with the same length and direction are two pictures of the same vector.

Vector addition and subtraction follow parallelogram laws.

The norm (magnitude) of $\langle a, b, c \rangle$ is

$$| \langle a, b, c \rangle | = \sqrt{a^2 + b^2 + c^2}$$

Direction is generally given by a unit vector, a vector whose norm is 1.

Parallel vectors have the same or opposite directions; each is a scalar multiple of the other.

Theorem: If $\vec{v}$ is a nonzero vector, the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v}.$$

Standard basis for $\mathbb{R}^3$:

$$\{\hat{\imath}, \hat{\jmath}, , \hat{k}\}$$

$$\hat{\imath} = \langle 1, 0, 0 \rangle$$

$$\hat{\jmath} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle$$

$$\langle a, b, c \rangle = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

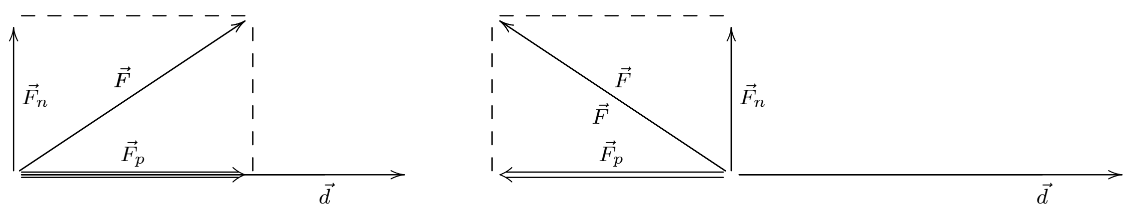
Notation: Sometimes vectors are written with an arrow on top, $\vec{v}$. Sometimes, instead, they are written in boldface, $\mathbf{v}$.

Sometimes the norm of the vector $\vec{v}$ is written $|\vec{v}|$ instead of $|\vec{v}|$.

It is common, particularly in physics and engineering, to write the vector $\langle a, b, c \rangle$ as $a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, or as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, or as $a\hat{x} + b\hat{y} + c\hat{z}$.

You may know from physics, "work equals force times distance." However, it isn't quite that simple. If the force acts in the opposite direction to the motion, it does negative work. (That is, work is done against the force.) If the force is perpendicular to the direction of motion, it does no work at all. If the force is at an angle to the direction of motion, the work depends on the component of the force parallel to the direction of motion.

Suppose an object acted on by force $\vec{F}$ moves in a straight line along vector $\vec{d}$. (There are also other forces acting on the object.) We are interested in the work done by $\vec{F}$. Here are two possible pictures.



The force $\vec{F}$ can be expressed as the sum of two components, $\vec{F}_p$ parallel to the direction of motion, and $\vec{F}_n$ normal (or perpendicular, or orthogonal) to the direction of motion. Only $\vec{F}_p$ does work on the moving object.

The vector $\vec{F}_p$ is called the *vector projection* of $\vec{F}$ on $\vec{d}$ (or in the direction of $\vec{d}$). It is sometimes called simply the projection of $\vec{F}$ on $\vec{d}$.

The work done by $\vec{F}$ on our moving object is

$$W = \begin{cases} |\vec{F}_p|\,|\vec{d}| & \text{if } \vec{F}_p \text{ has the same direction as } \vec{d}; \\ \\ -|\vec{F}_p|\,|\vec{d}| & \text{if } \vec{F}_p \text{ has the opposite direction from } \vec{d}. \end{cases}$$

The *scalar projection* of $\vec{F}$ on $\vec{d}$, sometimes denoted $proj_{\vec{d}}(\vec{F})$, is the quantity

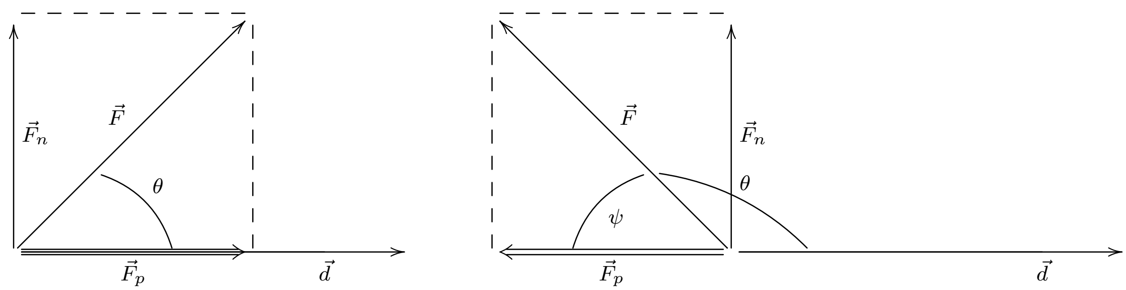
$$\begin{cases} |\vec{F}_p| & \text{if } \vec{F}_p \text{ has the same direction as } \vec{d}; \\ \\ -|\vec{F}_p| & \text{if } \vec{F}_p \text{ has the opposite direction from } \vec{d}. \end{cases}$$

$$W = \left(proj_{\vec{d}}(\vec{F})\right)|\vec{d}|$$

It is sometimes called the component, or coordinate, of $\vec{F}$ in the direction of $\vec{d}$.

"Work equals (scalar projection of force in the direction of motion) times distance."

Finding vector and scalar projections:



If $\vec{F}$ makes an angle of θ with $\vec{d}$, then the scalar projection of $\vec{F}$ onto $\vec{d}$ is

$$proj_{\vec{d}}(\vec{F}) = |\vec{F}|\cos(\theta)$$

and in our case, the work done by $\vec{F}$ on our object is the scalar projection of $\vec{F}$ in the direction of motion times the distance moved,

$$W = proj_{\vec{d}}(\vec{F})|\vec{d}| = |\vec{F}|\,|\vec{d}|\,\cos(\theta).$$

The vector projection of $\vec{F}$ onto $\vec{d}$ has the same direction as $\vec{d}$ if the scalar projection is positive, and the opposite direction if the scalar projection is negative.

Let $\vec{u}$ be a unit vector in the direction of $\vec{d}$,

$$\vec{u} = \frac{1}{|\vec{d}|}\,\vec{d}.$$

Then the vector projection of $\vec{F}$ onto $\vec{d}$ his

$$\overrightarrow{proj}_{\vec{d}}(\vec{F}) = \Big(proj_{\vec{d}}(\vec{F})\Big)\vec{u} = \Big(|\vec{F}|\cos(\theta)\Big)\vec{u} = \left(\frac{|\vec{F}|\cos(\theta)}{|\vec{d}|}\right)\,\vec{d}$$

There is a kind of product of vectors that helps us to compute these projections. The *dot product* of two vectors is a scalar, defined by

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|\,\cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. (We'll see an algebraic formula for the dot product shortly.)

The dot product is sometimes called the *scalar product*, because the dot product of two vectors is a scalar. This is not the same as scalar multiplication.

Example: If $\vec{v}$ and $\vec{w}$ are perpendicular to each other, then

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\left(\frac{\pi}{2}\right) = 0.$$

If $\vec{v}$ and $\vec{w}$ have the same direction,

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos(0) = |\vec{v}|\,|\vec{w}|,$$

and if they have opposite directions,

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos(\pi) = -|\vec{v}|\,|\vec{w}|.$$

In particular,

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2.$$

We can use the dot product to rewrite our formulas:

If $\vec{F}$ makes an angle of θ with $\vec{d}$, then the scalar projection of $\vec{F}$ onto $\vec{d}$ is

$$proj_{\vec{d}}(\vec{F}) = |\vec{F}| \cos(\theta) = \frac{|\vec{F}|\,|\vec{d}| \cos(\theta)}{|\vec{d}|} = \boxed{\frac{\vec{F} \cdot \vec{d}}{|\vec{d}|}}$$

and the vector projection of $\vec{F}$ onto $\vec{d}$ is

$$\overrightarrow{proj}_{\vec{d}}(\vec{F}) = \left(\frac{|\vec{F}| \cos(\theta)}{|\vec{d}|}\right) \vec{d} = \left(\frac{|\vec{F}|\,|\vec{d}| \cos(\theta)}{|\vec{d}|^2}\right) \vec{d} = \boxed{\left(\frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right) \vec{d}}$$

The work done by force $\vec{F}$ on an object moving in a straight line with displacement $\vec{d}$ is

$$\boxed{W = \vec{F} \cdot \vec{d}}$$

"Work equals force dot displacement."

Algebra of the dot product:

$$\boxed{\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

See the textbook for a proof of this formula using the Law of Cosines. In particular,

$$\langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = |\, \langle a_1, a_2, a_3 \rangle \,|^2.$$

The corresponding formulas work in $\mathbb{R}^2$ (and in $\mathbb{R}^n$) as well.

Theorem (basic facts about dot products):

$$\vec{v} \cdot \vec{w} = |\vec{v}|\, |\vec{w}| \cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

$$\vec{v} \cdot (\vec{w} + \vec{u}) = (\vec{v} \cdot \vec{w}) + (\vec{v} \cdot \vec{u})$$
$$\vec{v} \cdot (\vec{w} - \vec{u}) = (\vec{v} \cdot \vec{w}) - (\vec{v} \cdot \vec{u})$$
$$(t\vec{v}) \cdot \vec{w} = t(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (t\vec{w})$$
$$\vec{0} \cdot \vec{v} = 0$$
$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

Questions:

We know that, if $\vec{v}$ and $\vec{w}$ are nonzero vectors, then $\vec{v} \cdot \vec{w} = 0$ means that $\vec{v}$ and $\vec{w}$ are orthogonal (perpendicular to each other). What does $\vec{v} \cdot \vec{w} > 0$ mean geometrically?

The angle between $\vec{v}$ and $\vec{w}$ is acute.

The first theorem on this page includes a commutative law for dot products:

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

Is there an associative law for dot products, $(\vec{v} \cdot (\vec{u} \cdot \vec{w}) = (\vec{v} \cdot \vec{u}) \cdot \vec{w})$? Why or why not?

No. $(\vec{u} \cdot \vec{w})$ is a number, and you cannot take the dot product of a vector and a number, so the expression $\vec{v} \cdot (\vec{u} \cdot \vec{w})$ doesn't make sense.

Example: Find the vector and scalar projections of the vector $\vec{v} = \langle -1, -1, -2 \rangle$ in the direction of the vector $\vec{w} = \langle 3, 4, 12 \rangle$.

$$\vec{v} \cdot \vec{w} = (-1)(3) + (-1)(4) + (-2)(12) = -31$$

$$proj_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \frac{-31}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{-31}{13}.$$

$$\overrightarrow{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2}\, \vec{w} = \frac{-31}{169} \langle 3, 4, 12 \rangle = \left\langle \frac{-93}{169}, \frac{-124}{169}, \frac{-372}{169} \right\rangle.$$

Example: TRUE or FALSE?

> If $\vec{u}$ is a unit vector, then the scalar projection of $\vec{v}$ on $\vec{u}$ is just $\vec{v} \cdot \vec{u}$, and the vector projection of $\vec{v}$ on $\vec{u}$ is just $(\vec{v} \cdot \vec{u})\, \vec{u}$.

Explain.

True in both cases. Since $|\vec{u}| = 1$:

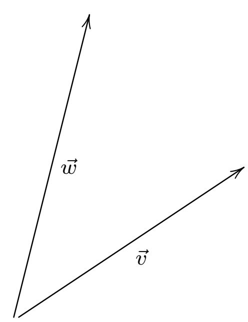
$$proj_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|} = \frac{\vec{v} \cdot \vec{u}}{1} = \vec{v} \cdot \vec{u};$$

$$\overrightarrow{proj}_{\vec{w}}(\vec{u}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{u}|^2}\, \vec{u} = \frac{\vec{v} \cdot \vec{w}}{1}\, \vec{u} = (\vec{v} \cdot \vec{u})\, \vec{u}.$$

Definition: The cross product, or vector product, of vectors $\vec{v}$ and $\vec{w}$ is the vector $\vec{v} \times \vec{w}$ with the following properties:

1. $|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin(\theta)$ where θ is the angle between $\vec{v}$ and $\vec{w}$.
2. $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.
3. $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$ are oriented according to the right-hand rule: If all three vectors are drawn from the same point, and you are looking down from the top of $\vec{v} \times \vec{w}$, rotating from $\vec{v}$ around to $\vec{w}$ appears as a counterclockwise rotation.

Note: The cross product is defined only in $\mathbb{R}^3$.



$\vec{v} \times \vec{w}$ points out of the paper. $\vec{w} \times \vec{v}$ points into the paper.

Three other ways to remember the right-hand rule:

Point the thumb of your right hand in the direction of $\vec{v} \times \vec{w}$. The fingers curl in the direction of rotation from $\vec{v}$ to $\vec{w}$.

Hold your arms out parallel to the ground and pointing slightly forwards, at an angle to each other. If your right hand points in the direction of the first vector $\vec{v}$ and your left hand points in the direction of the second vector $\vec{w}$, then your head points in the direction of the vector $\vec{v} \times \vec{w}$. (Provided, of course, that you haven't crossed your arms.)

The vectors $\vec{v}$, $\vec{w}$, and $\vec{v} \times \vec{w}$, in that order, are oriented in the same way as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, in that order. And, in fact, $\hat{\imath} \times \hat{\jmath} = \hat{k}$.

Example: Find the vector $\hat{k} \times \hat{\jmath}$.

Since $\hat{k}$ and $\hat{k}$ are unit vectors and the sine of the angle between them is 1, by the geometric interpretation of the cross product, $\hat{k} \times \hat{\jmath}$ must be a unit vector perpendicular to them both. That is, it must be either $\hat{\imath}$ or $-\hat{\imath}$. Using the right-hand rule, we see $\hat{k} \times \hat{\jmath} = -\hat{\imath}$.

Algebra of the cross product.

The determinant of a matrix will help us compute cross products without getting too mixed up.

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}}_{***} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

*** is the determinant of the matrix left when you cross out the row and column of a_1.
Notice the alternating + and − signs.

The matrix determinant has many useful applications. We're going to use it in the formula for cross product:

$$\langle v_1, v_2, v_3\rangle \times \langle w_1, w_2, w_3\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \hat{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \hat{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \hat{k} =$$

$$\langle v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1\rangle .$$

Example:

$$\hat{k} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \hat{k} =$$

$$((0)(0) - (1)(1))\hat{i} - ((0)(0) - (0)(1))\hat{j} + ((0)(1) - (0)(0))\hat{k} = -\hat{i}$$

Example: Compute $\langle 1, 2, 1\rangle \times \langle 1, 0, -1\rangle$. Use the dot product to check that the cross product is orthogonal to both factors.

$$\langle 1, 2, 1\rangle \times \langle 1, 0, -1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \hat{k} =$$

$$(-2)\hat{i} - (-2)\hat{j} + (-2)\hat{k} = \langle -2,\ 2,\ -2\rangle .$$

Check: $\langle 1, 2, 1\rangle \cdot \langle -2, 2, -2\rangle = -2 + 4 - 2 = 0$ and $\langle 1, 0, -1\rangle \cdot \langle -2, 2, -2\rangle = -2 + 0 + 2 = 0$.

Theorem: $|\vec{v} \times \vec{w}|$ is the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$.

Theorem:

$$\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$$

$$t(\vec{v} \times \vec{w}) = t\vec{v} \times \vec{w} = \vec{v} \times t\vec{w}$$

$$\vec{v} \times (\vec{w} + \vec{u}) = (\vec{v} \times \vec{w}) + (\vec{v} \times \vec{u})$$

$$\vec{v} \times (\vec{w} \times \vec{u}) = (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}$$

Warning; The cross product is NOT commutative and NOT associative.

Definition: The *triple product* of $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$, and $\vec{u} = \langle u_1, u_2, u_3 \rangle$, in that order, is

$$\vec{v} \cdot (\vec{w} \times \vec{u}) = (\vec{v} \times \vec{w}) \cdot \vec{u} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Theorem: The absolute value of the triple product of $\vec{v}$, $\vec{w}$, and $\vec{u}$ is the volume of the parallelepiped with edges $\vec{v}$, $\vec{w}$, and $\vec{u}$.

The triple product is positive if $\vec{v}$, $\vec{w}$, and $\vec{u}$, in that order, are oriented according to the right hand rule in the same way as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ (or as $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$). It is negative if they have the opposite orientation.

(This is related to some of the ways in which determinants are useful.)

Example: What does it mean geometrically if $\vec{v} \times \vec{w} = \vec{0}$?

$\vec{v}$ and $\vec{w}$ are parallel.

If $|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}|$?

$\vec{v}$ and $\vec{w}$ are perpendicular.

If $\vec{v} \cdot (\vec{w} \times \vec{u}) = 0$?

$\vec{v}$, $\vec{w}$, and $\vec{u}$ are coplanar.

If $|\vec{v} \cdot (\vec{w} \times \vec{u})| = |\vec{v}|\,|\vec{w}|\,|\vec{u}|$?

$\vec{v}$, $\vec{w}$, and $\vec{u}$ are pairwise orthogonal (perpendicular).

Is it always true that $|\vec{v} \cdot (\vec{w} \times \vec{u})| \leq |\vec{v}|\,|\vec{w}|\,|\vec{u}|$? Why or why not?

Yes, since

$$|\vec{v} \cdot (\vec{w} \times \vec{u})| = |\vec{v}||\vec{w} \times \vec{u}||\cos\theta| = |\vec{v}|(|\vec{w}||\vec{u}|\sin\varphi)|\cos\theta|$$

where θ is the angle between $\vec{v}$ and $\vec{w} \times \vec{u}$ and φ is the angle between $\vec{w}$ and $\vec{u}$.

A plane contains the triangle with vertices $P = (1, 1, 1)$, $Q = (1, 2, 3)$, and $R = (2, 2, -1)$. Find two vectors parallel to the plane but not parallel to each other.

$$\begin{aligned}\overrightarrow{PR} &= \langle 2-1, 2-1, -1-1\rangle = \langle 1, 1, -2\rangle\,;\\ \overrightarrow{QR} &= \langle 2-1, 2-2, -1-3\rangle = \langle 1, 0, -4\rangle\,.\end{aligned}$$

Find a vector perpendicular to the plane. (Hint: Use the cross product.)

(We will see next time that if you know a point on a plane and a vector perpendicular to the plane, you can write down the equation of the plane, so this is a useful thing to be able to do.)

$$\langle 1, 1, -2\rangle \times \langle 1, 0, -4\rangle = \langle -4,\ 2,\ -1\rangle\,.$$

Find the volume of the parallelepiped whose edges are $\vec{v} = \langle 1, 2, 1\rangle$, $\vec{w} = \langle -1, 0, 1\rangle$, $\vec{u} = \langle 1, 1, 2\rangle$.

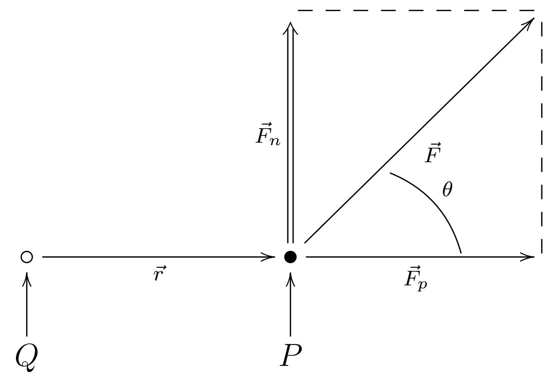
$$\vec{v}\cdot(\vec{u}\times\vec{w}) = \begin{vmatrix} 1 & 2 & 1\\ -1 & 0 & 1\\ 1 & 1 & 2\end{vmatrix} = 1\begin{vmatrix} 0 & 1\\ 1 & 2\end{vmatrix} - 2\begin{vmatrix} -1 & 1\\ 1 & 2\end{vmatrix} + 1\begin{vmatrix} -1 & 0\\ 1 & 1\end{vmatrix} = 1(-1) - 2(-3) + 1(-1) = 4.$$

The volume is $|\vec{v}\cdot(\vec{u}\times\vec{w})| = 4$.

Use the algebraic rules for dot products and cross products (for example, the distributive law $\vec{v}\times(\vec{w}+\vec{u}) = (\vec{v}\times\vec{w}) + (\vec{v}+\vec{u})$) to show that the triple product of $\vec{v}$, $\vec{w}$, and $s\vec{v}+t\vec{w}$ is always zero, for any vectors $\vec{v}$ and $\vec{w}$ and any scalars s and t.

Now use geometric reasoning to show the same thing.

Physics Connection:



An object • (at point P) is attached by a rigid rod to a fixed point ∘ (point Q), but is free to rotate around that point in any direction. The vector $\vec{r}$ goes to the object from the fixed point around which it may rotate. A force $\vec{F}$ acts on the object.

The torque vector $\vec{\tau}$ represents the tendency of the object to rotate around the fixed point, caused by the force $\vec{F}$.

If we decompose $\vec{F}$ into two component forces, $\vec{F}_p$ parallel to $\vec{r}$ and $\vec{F}_n$ normal to $\vec{r}$, only $\vec{F}_n$ imparts torque.

The magnitude of the torque depends both on the force and on the distance from the fixed point Q (think levers, or seesaws), and is

$$|\tau| = |\vec{r}|\,|\vec{F}_n| = |\vec{r}|\,|\vec{F}|\sin(\theta) = |\vec{r} \times \vec{F}|.$$

The direction of τ gives the direction of the axis around which the object rotates.

> Repetition for emphasis: The direction of τ gives the **direction of the axis** around which the object rotates. It does *not* give the direction in which the object moves.
>
> This is the way we represent rotational motion. As the earth rotates around its axis, different points on its surface are moving in different directions, but the axis of rotation is the same for the entire globe. So torque, rotational (angular) velocity, etc. are represented by vectors pointing along the axis of rotation, which in the case of the earth is the line through the north and south poles.

In this picture, since $\vec{r}$ and $\vec{F}$ are both in the plane of the paper, the axis of rotation is perpendicular to the paper, so $\vec{\tau}$ points in a direction perpendicular to the paper — either out or in. By convention, since the rotation is counterclockwise as we look at the paper, the torque vector τ points out of the paper toward us.

In other words, using the right-hand rule, we see the torque τ is given by

$$\boxed{\tau = \vec{r} \times \vec{F}.}$$