Math 11
Fall 2016
Section 1
Wednesday, October 26, 2016

First, some important points from the last class:

Vector Fields:

Definition: An n-dimensional *vector field* is a function $F : \mathbb{R}^n \to \mathbb{R}^n$. We think of it as assigning to each point in n-dimensional space a vector in n-dimensional space.

Examples include gradient fields, fluid flow fields, gravitational fields, and electric fields.

Definition: A vector field F is called *conservative* if it is a gradient field. If we write $F = -\nabla f$, then f is a *potential function* for f.

Line Integrals:

$$\int_\gamma f\,ds = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(\vec{r}^{\,*}_i)\Delta s_i\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(\vec{r}(t_i^*))\,|\vec{r}\,'(t_i^*)|\,\Delta t\right) = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt.$$

$$ds = \frac{ds}{dt}\,dt = |\vec{r}\,'(t)|\,dt.$$

$\displaystyle\int_\gamma 1\,ds =$ arc length of γ; $\quad \displaystyle\frac{1}{\text{arc length of }\gamma}\int_\gamma f\,ds =$ average value of f on γ;

$\displaystyle\int_\gamma$ linear mass density (grams per meter) $ds =$ mass of γ;

$\displaystyle\int_\gamma f\,ds =$ area of fence with base γ and height given by f.

Suppose $F(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a vector function on $\mathbb{R}^2$, and γ is a smooth *oriented* curve in $\mathbb{R}^2$ parametrized by $\vec{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$.

$$ds = |\vec{r}\,'(t)|\,dt = |\langle x'(t), y'(t)\rangle|\,dt$$

$$\int_\gamma F\cdot\vec{T}\,ds = \int_a^b \langle P(x(t),y(t)), Q(x(t),y(t))\rangle \cdot \langle x'(t), y'(t)\rangle\ dt =$$

$$\int_\gamma P\,dx + Q\,dy = \int_\gamma P\,dx + \int_\gamma Q\,dy,$$

$$\int_\gamma P\,dx = \int_a^b P(x(t),y(t))\,x'(t)\,dt \qquad \int_\gamma Q\,dy = \int_a^b Q(x(t),y(t))\,y'(t)\,dt.$$

$$dx = x'(t)\,dt \qquad dy = y'(t)\,dt \qquad \vec{T}\,ds = \langle dx, dy\rangle = \vec{r}\,'(t)\,dt.$$

Example: Suppose $F(x, y, z) = \langle x, y, z\rangle$ and γ is the portion of the helix parametrized by $\vec{r}(t) = \langle \cos(t), \sin(t), \rangle$ for $0 \leq t \leq 2\pi$. Find

$$\int_\gamma F \cdot \vec{T}\, ds.$$

$$\vec{T}\, ds = \langle dx, dy, dz\rangle\,.$$

$$\int_\gamma F \cdot \vec{T}\, ds = \int_\gamma \langle x, y, z\rangle \cdot \langle dx, dy, dz\rangle = \int_\gamma x\, dx + y\, dy + z\, dz = \int_\gamma x\, dx + \int_\gamma y\, dy + \int_\gamma z\, dz.$$

$$\int_\gamma x\, dx = \int_1^{-1} x\, dx + \int_{-1}^{1} x\, dx = 0.$$

$$\int_\gamma y\, dy = \int_0^{2\pi} (\sin(t))(\cos(t)\, dt) = \left.\frac{\sin^2(t)}{2}\right|_{t=0}^{t=2\pi} = 0.$$

$$\int_\gamma z\, dz = \int_0^{2\pi} z\, dz = \left.\frac{z^2}{2}\right|_{z=0}^{z=2\pi} = 2\pi^2.$$

OR

$$\int_\gamma F \cdot \vec{T}\, ds = \int_\gamma \langle x, y, z\rangle \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} \langle \cos(t), \sin(t), t\rangle \cdot \langle -\sin(t), \cos(t), 1\rangle\; dt$$

$$= \int_0^{2\pi} t\, dt = \left.\frac{t^2}{2}\right|_{t=0}^{t=2\pi} = 2\pi^2.$$

Another piece of notation: We sometimes write this integral as

$$\int_\gamma F \cdot \vec{T}\, ds = \int_\gamma F \cdot d\vec{r}.$$

$$d\vec{r} = \vec{r}\,'(t)\, dt = \left(\frac{1}{|r\,'(t)|} r\,'(t)\right)\, |r\,'(t)|\, dt = \vec{T}\, ds.$$

Interpreting this integral:

First, suppose $F = \nabla f$. Then $F \cdot \vec{T}$ is the rate of change of f (with respect to distance) in the direction of γ. Since the line integral is an integral with respect to distance, integrating the rate of change of f should give the net change in f.

$$\int_\gamma \nabla f \cdot \vec{T}\, ds \text{ should equal the net change in } f \text{ along } \gamma.$$

It does, and we will be able to prove that using the Chain Rule and the single variable Fundamental Theorem of Calculus.

For now, we can check that it is true in the above example:

$$F(x, y, z) = \langle x, y, z \rangle = \nabla f(x, y, z) \quad \text{where} \quad f(x, y, z) = \frac{x^2 + y^2 + z^2}{2}.$$

$$\int_\gamma F \cdot \vec{T}\, ds = 2\pi^2.$$

Since γ goes from $(1, 0, 0)$ at $t = 0$ to $(1, 0, 2\pi)$ at $t = 2\pi$, the net change in f along γ is

$$f(1, 0, 2\pi) - f(1, 0, 0) = \frac{1 + 4\pi^2}{2} - \frac{1}{2} = 2\pi^2.$$

Now, instead, suppose that F is a force field, so $F(x, y, z)$ is the force exerted on an object when it is at point (x, y, z). If our object is moving along γ with position function $\vec{r}(t)$, then for a small time interval of length Δt containing time t, we have approximately

$$F(\vec{r}(t)) \cdot (\vec{r}\,'(t)\, \Delta t) = (\text{force}) \cdot (\text{displacement}) = \text{work done by force during time interval.}$$

If F is a force field, so $F(x, y, z)$ is the force exerted on an object when it is at point (x, y, z), and an object moves along the oriented curve γ, then the work done by that force on that object is

$$W = \int_\gamma F \cdot d\vec{r}.$$

Orientation matters here. If we reverse the direction of motion, we change the sign of W.

However, parametrization does not matter. Any two parametrizations of γ with the same orientation give the same value for the line integral $\displaystyle\int_\gamma F \cdot d\vec{r}$.

Some mathy language: The line integral of a vector field F along an oriented curve γ is *independent of* parametrization. It does, however, depend on orientation. The line integral of a scalar function f along a curve γ is independent of both orientation and parametrization.

Example: Find the work done by the force $F(x,y,z) = \langle -y, x, z^2 \rangle$ on an object moving around the circle $x^2 + y^2 = 1$ in the plane $z = 1$, oriented clockwise as seen from above.

$$\vec{r}(t) = \langle \cos(t), -\sin(t), 1 \rangle \qquad 0 \le t \le 2\pi.$$

$$\int_\gamma F \cdot d\vec{r} = \int_0^{2\pi} \langle \sin(t), \cos(t), 1 \rangle \cdot \langle -\sin(t), -\cos(t), 0 \rangle \; dt = \int_0^{2\pi} (-1)\, dt = -2\pi.$$

The Fundamental Theorem of Line Integrals: Suppose γ is a smooth curve, and f a function with continuous derivative. Then

$$\int_\gamma \nabla f \cdot d\vec{r} = f(\text{end}(\gamma)) - f(\text{start}(\gamma)) = \text{ net change in } f \text{ along } \gamma.$$

Proof: Parametrize γ by $\vec{r}(t)$ with $a \le t \le b$.

$$\int_\gamma \nabla f \cdot d\vec{r} = \int_a^b \underbrace{\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\frac{d}{dt}(f(\vec{r}(t))} \, dt =$$

$$f(\vec{r}(t)\Big|_{t=a}^{t=b} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(\text{end}(\gamma)) - f(\text{start}(\gamma)).$$

Example: Find the work done by the vector field

$$F(x,y,z) = \left\langle \frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$$

on an object moving along γ, the portion of the helix parametrized by $\vec{r}(t) = \left\langle \cos(t), \sin(t), \frac{t}{\pi} \right\rangle$ for $0 \le t \le 2\pi$.

Since $F = \nabla f$ where $f(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$, we have

$$\int_\gamma F \cdot d\vec{r} = \int_\gamma \nabla f \cdot d\vec{r} = f(\vec{r}(2\pi)) - f(\vec{r}(0)) = f(1,0,2) - f(1,0,0) = \frac{1}{\sqrt{5}} - \frac{1}{1} = \frac{\sqrt{5}-5}{5}.$$

In that last example,

$$F(x,y,z) = \left\langle \frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right\rangle$$

is pointing directly toward the origin, and has magnitude

$$\frac{1}{x^2+y^2+z^2} = \frac{1}{(\text{distance from origin})^2}.$$

This is a constant multiple of the gravitational force between a particle at position (x, y, z) and a mass located at the origin. We have $F = \nabla f$, and the potential function $-f$ is given by

$$-f(x,y,z) = -(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{-1}{\text{distance from origin}}.$$

We can think of this as the gravitational potential, or potential energy, of the particle when it is located at position (x, y, z). As the particle moves along γ, the work done by the force of gravity is

$$W = \int_\gamma \nabla f \cdot d\vec{r} = f(\text{end}(\gamma)) - f(\text{start}(\gamma)).$$

This is the net increase in f along γ, or (since the potential function is $-f$) the net decrease in potential along γ. That is

$$\text{work done by } F \ = -(\text{change in potential}).$$

In this example, the farther from the origin, the greater the potential.

The particle moving along the helix ended up farther from the origin than it started. The work done by this force was negative, and the change in potential was positive.

To see how we could have found that potential function, we'll do the slightly less complicated two-dimensional version.

Example Suppose

$$F(x,y) = \left\langle \frac{-x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{-y}{(x^2+y^2)^{\frac{3}{2}}} \right\rangle.$$

Find a potential function for F.

We want to write $F = \nabla f$; then the potential function will be $-f$. So we want

$$f_x(x,y) = \frac{-x}{(x^2+y^2)^{\frac{3}{2}}} \qquad f_y(x,y) = \frac{-y}{(x^2+y^2)^{\frac{3}{2}}}$$

Integrating the first equation with respect to x:

$$f(x,y) = \int (-x)(x^2+y^2)^{-\frac{3}{2}}\,dx = (x^2+y^2)^{-\frac{1}{2}} + g(y).$$

Differentiating with respect to y:

$$f_y(x,y) = (-y)(x^2+y^2)^{-\frac{3}{2}} + g'(y).$$

Comparing with our earlier expression for f_y, we see

$$g'(y) = 0.$$

That is, $g(y)$ is constant. Since all we want is one potential function, we can take $g(y) = 0$, and

$$f(x,y) = (x^2+y^2)^{-\frac{1}{2}};$$

$$\text{potential } = -f(x,y) = -(x^2+y^2)^{-\frac{1}{2}}.$$

If we find it unsatisfying to have negative potential, we can add a very large constant C without changing the gradient. Now we have

$$\text{potential } = C - (x^2+y^2)^{-\frac{1}{2}}.$$

We still get negative potential at points extremely close to the origin. Negative potential energy may not make physical sense, but a large mass concentrated at a single, dimensionless point does not make physical sense either. If we instead consider the mass to be distributed throughout a tiny ball centered at the origin, we can adjust C so the potential at the surface of the ball is zero, which is much more satisfying.

Example: Show that $F(x, y) = \langle -y, x\rangle$ is not conservative.

If we had $F = \nabla f$, we would have

$$f_x(x, y) = -y \qquad f_y(x, y) = x$$

$$f_{xy}(x, y) = -1 \qquad f_{yx}(x, y) = 1,$$

which contradicts Clairaut's Theorem.

Suppose we tried to find a potential function. We would get:

$$f_x(x, y) = -y \qquad \text{and so} \qquad f(x, y) = -xy + g(y)$$

$$f_y(x, y) = -x + g'(y) \qquad \text{and so} \qquad -x + g'(y) = x$$

This would mean $g'(y) = 2x$, but $2x$ is not a function of y. This tells us our task is impossible; F is not conservative.

Example: The vector field

$$F(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

is not conservative on all of $\mathbb{R}^2$, but it is conservative on the half plane $x > 0$. Suppose that γ is the straight line from $(1,0)$ to $(3,3)$. Without finding a potential function for F, find

$$\int_\gamma F \cdot d\vec{r}.$$

We know $F = \nabla f$ for some unknown f, for which we have

$$\int_\gamma F \cdot d\vec{r} = f(3,3) - f(1,0).$$

Suppose that ψ is the curve that goes in a straight line from $(1,0)$ to $(3\sqrt{2},0)$, and then along a circle around the origin to $(3,3)$. Then

$$\int_\psi F \cdot d\vec{r} = f(3,3) - f(1,0) = \int_\gamma F \cdot d\vec{r}.$$

It is easier to integrate F along ψ. Let ψ_1 be the straight line from $(1,0)$ to $(3\sqrt{2},0)$, and ψ_2 be the circular arc from $(3\sqrt{2},0)$ to $(3,3)$. Then

$$\int_\psi F \cdot d\vec{r} = \int_{\psi_1} F \cdot d\vec{r} + \int_{\psi_2} F \cdot d\vec{r} = \int_{\psi_1} F \cdot \vec{T}\, ds + \int_{\psi_2} F \cdot \vec{T}\, ds.$$

Along ψ_1 we have $T = \langle 1, 0 \rangle$ and $y = 0$, so $F \cdot \vec{T} = 0$. The line integral of F along ψ_1 is 0.

Along ψ_2, we can see T and F have the same direction, and since T is a unit vector, we have

$$F \cdot \vec{T} = |F|\,|\vec{T}| \cos\theta = |F| = \left| \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \right| = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{3\sqrt{2}}.$$

This is a constant, and integrated along ψ_2 with respect to arc length, it gives $\dfrac{1}{3\sqrt{2}}$ times the arc length of γ. Since γ is one eighth of a circle of radius $3\sqrt{2}$, its arc length is $\dfrac{\pi 3\sqrt{2}}{4}$ and the integral is $\dfrac{\pi}{4}$. Putting it all together:

$$\int_\gamma F \cdot d\vec{r} = \int_\psi F \cdot d\vec{r} = \int_{\psi_1} F \cdot \vec{T}\, ds + \int_{\psi_2} F \cdot \vec{T}\, ds = 0 + \frac{\pi}{4} = \frac{\pi}{4}.$$

Example: In general, if γ and ψ are two different smooth curves starting and ending at the same points, and F is a vector field that is not conservative, then

$$\int_\gamma F \cdot \vec{T}\, ds \neq \int_\psi F \cdot \vec{T}\, ds.$$

Show this by evaluating both integrals when γ is the portion of the unit circle from $(1, 0)$ counterclockwise to $(-1, 0)$, and ψ is the straight line segment from $(1, 0)$ to $(-1, 0)$, and $F(x, y) = \langle -y, x\rangle$.

For practice, try evaluating each line integral in three different ways, first by parametrizing the curve and evaluating the line integral as $\displaystyle\int F \cdot d\vec{r}$, second by evaluating separately both pieces of the line integral in the form $\displaystyle\int P\, dx + Q\, dy$, and third by figuring out what $F \cdot \vec{T}$ is on the curve and using that to evaluate the line integral as $\displaystyle\int F \cdot \vec{T}\, ds$.

Example: A force field is given by

$$F(x, y, z) = \langle y, 2x, y \rangle$$

and γ is the intersection of the surfaces $y = x^2$ and $z = x^3$.

Show that F is not conservative.

Find the work done by F on an object that moves along γ from $(0, 0, 0)$ to $(1, 1, 1)$.

To remember what's going with the notation:
Parametrize γ by $\langle x, y \rangle = \vec{r}(t)$. Then:

$$ds = |\vec{r}\,'(t)|\, dt \qquad dx = x'(t)\, dt \qquad dy = y'(t)\, dt.$$

These differentials are scalars. The following differentials are vectors:

$$d\vec{r} = \vec{r}\,'(t)\, dt = \langle x'(t), y'(t) \rangle \; dt = \langle x'(t)\, dt,\; y'(t)\, dy \rangle = \langle dx,\; dy \rangle\, .$$

$$d\vec{r} = \vec{r}\,'(t)\, dt = \left(\frac{1}{|\vec{r}\,'(t)|} \vec{r}'(t) \right) (|\vec{r}'(t)|\, dt) = \vec{T}\, ds.$$

We can think of ds as the arc length of the infinitely small portion of γ traversed over an infinitely small time period of length dt, and of $d\vec{r}$ as the infinitely small displacement over that same time period.