Math 11
Fall 2016
Section 1
Friday, October 28, 2016

First, some important points from the last class (and a few from the one before):

Definition: An n-dimensional *vector field* is a function $F : \mathbb{R}^n \to \mathbb{R}^n$. We think of it as assigning to each point in n-dimensional space a vector in n-dimensional space. Our go-to examples will generally be force fields, fluid flow fields, and gradient fields.

A vector field F is called *conservative* if it is a gradient field. If we write $F = -\nabla f$, then f is a *potential function* for f.

If F is a force field, so $F(x, y, z)$ is the force exerted on an object when it is at point (x, y, z), and an object moves along the oriented curve γ, then the work done by that force on that object is the line integral of F along γ, $W = \int_\gamma F \cdot d\vec{r}$. This integral is dependent on orientation but independent of parametrization. The line integral $\int_\gamma f\ ds$ of a scalar function f along γ is independent of both orientation and parametrization.

If $\vec{r}(t) = \langle x(t), y(t) \rangle$ parametrizes γ, then on γ:

$$\boxed{\text{element of arc length } = ds = |\vec{r}\,'(t)|\, dt \qquad dx = x'(t)\, dt \qquad dy = y'(t)\, dt}.$$

These differentials are scalars. The following differentials are vectors:

$$\boxed{\text{element of displacement } = d\vec{r} = \vec{r}\,'(t)\, dt = \langle x'(t), y'(t) \rangle\ dt = \langle x'(t)\, dt,\ y'(t)\, dy \rangle = \langle dx,\ dy \rangle}.$$

$$\boxed{d\vec{r} = \vec{r}\,'(t)\, dt = \left(\frac{1}{|\vec{r}\,'(t)|} \vec{r}'(t) \right) (|\vec{r}'(t)|\, dt) = \vec{T}\, ds}.$$

The Fundamental Theorem of Line Integrals: Suppose γ is a smooth curve, and f a function with continuous derivative. Then

$$\int_\gamma \nabla f \cdot d\vec{r} = f(\text{end}(\gamma)) - f(\text{start}(\gamma)) = \text{ net change in } f \text{ along } \gamma.$$

Example: Show that $F(x, y) = \langle -y, x \rangle$ is not conservative.

If we had $F = \nabla f$, we would have

$$f_x(x, y) = -y \qquad f_y(x, y) = x$$

$$f_{xy}(x, y) = -1 \qquad f_{yx}(x, y) = 1,$$

which contradicts Clairaut's Theorem.

Suppose we tried to find a potential function. We would get:

$$f_x(x, y) = -y \quad \text{and} \quad f_y(x, y) = x.$$

Differentiating the first equation with respect to x, remembering the constant of integration is a function of y:

$$f(x, y) = -xy + g(y).$$

Differentiating with respect to y:

$$f_y(x, y) = -x + g'(y) \quad \text{and so} \quad -x + g'(y) = x$$

This would mean $g'(y) = 2x$, but $2x$ is not a function of y. This tells us our task is impossible; F is not conservative.

Example: The vector field $F(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ is not conservative on $\mathbb{R}^2$, but it is conservative on the half plane $x > 0$. Suppose that γ is the straight line from $(1,0)$ to $(3,3)$. Without finding a potential function for F, find

$$\int_\gamma F \cdot d\vec{r}.$$

We know $F = \nabla f$ for some unknown f, for which we have

$$\int_\gamma F \cdot d\vec{r} = f(3,3) - f(1,0).$$

Suppose that ψ is the curve that goes in a straight line from $(1,0)$ to $(3\sqrt{2},0)$, and then along a circle around the origin to $(3,3)$. Then

$$\int_\psi F \cdot d\vec{r} = f(3,3) - f(1,0) = \int_\gamma F \cdot d\vec{r}.$$

It is easier to integrate F along ψ. Let ψ_1 be the straight line from $(1,0)$ to $(3\sqrt{2},0)$, and ψ_2 be the circular arc from $(3\sqrt{2},0)$ to $(3,3)$. Then

$$\int_\psi F \cdot d\vec{r} = \int_{\psi_1} F \cdot d\vec{r} + \int_{\psi_2} F \cdot d\vec{r} = \int_{\psi_1} F \cdot \vec{T}\, ds + \int_{\psi_2} F \cdot \vec{T}\, ds.$$

Along ψ_1 we have $T = \langle 1, 0\rangle$ and $y = 0$, so $F = \langle 0, x^{-1}\rangle$ and $F \cdot \vec{T} = 0$. The line integral of F along ψ_1 is 0.

Along ψ_2, we can see T and F have the same direction, and since T is a unit vector, we have

$$F \cdot \vec{T} = |F|\,|\vec{T}|\cos\theta = |F| = \left| \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \right| = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{3\sqrt{2}}.$$

This is a constant, and integrated along ψ_2 with respect to arc length, it gives $\frac{1}{3\sqrt{2}}$ times the arc length of γ. Since γ is one eighth of a circle of radius $3\sqrt{2}$, its arc length is $\frac{\pi 3\sqrt{2}}{4}$ and the integral is $\frac{\pi}{4}$. Putting it all together:

$$\int_\gamma F \cdot d\vec{r} = \int_\psi F \cdot d\vec{r} = \int_{\psi_1} F \cdot \vec{T}\, ds + \int_{\psi_2} F \cdot \vec{T}\, ds = 0 + \frac{\pi}{4} = \frac{\pi}{4}.$$

Note: You can check that for $x > 0$ we have $F = \nabla f$, where $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$. That is, $f = \theta$ (the θ of polar coordinates). The integral of F along γ is the net change in θ from $(1,0)$ to $(3,3)$. It makes sense that we can't express F as a gradient on $\mathbb{R}^2$ (or even on all of $\mathbb{R}^2$ except the origin), because there is no way to choose θ continuously.

Just for practice, let's show directly that if γ is the arc of a circle around the origin from $\theta = \theta_1$ to $\theta = \theta_2$, then the line integral of $F(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ along γ equals $\theta_2 - \theta_1$.

Let c be the radius of the circle. The circle has polar coordinate equation $r = c$ and the beginning and ending points of γ have polar coordinates (c, θ_1) and (c, θ_2). We can parametrize γ by $x = c\cos t$ and $y = c\sin t$ for $\theta_1 \le t \le \theta_2$. Then

$$\vec{r}(t) = \langle c\cos t, c\sin t\rangle \qquad \vec{r}\,'(t) = \langle -c\sin t, c\cos t\rangle \qquad F(\vec{r}(t)) = \left\langle \frac{-c\sin t}{c^2}, \frac{c\cos t}{c^2} \right\rangle$$

$$\int_\gamma F \cdot d\vec{r} = \int_{\theta_1}^{\theta_2} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_{\theta_1}^{\theta_2} \left\langle \frac{-c\sin t}{c^2}, \frac{c\cos t}{c^2} \right\rangle \cdot \langle -c\sin t, c\cos t\rangle\,dt =$$
$$\int_{\theta_1}^{\theta_2} \frac{c^2\sin^2(t) + c^2\cos^2(t)}{c^2}\,dt = \int_{\theta_1}^{\theta_2} 1\,dt = \theta_2 - \theta_1.$$

Notice that if we let $c = 1$, $\theta_1 = 0$ and $\theta_2 = 2\pi$, then γ is the unit circle oriented counterclockwise from $(1, 0)$ around to $(1, 0)$ again, and we get

$$\int_\gamma F \cdot d\vec{r} = 2\pi.$$

If we had $F = \nabla f$, the Fundamental Theorem of Line Integrals would tell us that

$$\int_\gamma F \cdot d\vec{r} = \int_\gamma \nabla f \cdot d\vec{r} = f(1, 0) - f(1, 0) = 0.$$

Since $2\pi \neq 0$, this tells us that F is not a conservative vector field on any open region containing the entire unit circle.

Note: We can check that $F = \nabla f$ would not violate Clairaut's Theorem for this function:

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2 - x^2}{x^2+y^2} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right).$$

Theorem: If the continuous vector field F is conservative (meaning $F = \nabla f$) on an open connected region D (connected means any two points in D can be connected via a curve in D), then:

1. $\displaystyle\int_\gamma F \cdot d\vec{r}$ is *path independent* on D. This means that if γ and ψ are two oriented curves in D with the same starting and ending points, then the line integrals of F along γ and along ψ are equal.

2. If γ is a smooth closed curve in D (closed means its end point equals its beginning point), then $\displaystyle\int_\gamma F \cdot d\vec{r} = 0$.

3. If the components of F have continuous partial derivatives, then F respects the Clairaut Theorem conditions: If $F = \langle P, Q \rangle$, then $P_y = Q_x$. If $F = \langle P, Q, R \rangle$, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.

Proof: Parts (1) and (2) follow from the Fundamental Theorem of Line Integrals, and part (3) from Clairaut's Theorem.

Theorem: If F is a continuous vector field on an open connected region D, then if either (1) or (2) holds, it follows that F is conservative on D.

Proof: We'll see this a little later.

Note: (1) $\iff$ (2) is not too hard to show, because if γ and ψ are two oriented curves in D with the same starting and ending points, then γ followed by $-\psi$ (meaning ψ in the opposite direction) is a closed curve.

Theorem: If F is a continuous vector field on an open, simply connected (no holes) region D in $\mathbb{R}^2$, the component functions of F have continuous partial derivatives, and F respects the Clairaut Theorem conditions, then F is conservative on D.

Proof: Deferred until after we have proven Green's Theorem.

Note: The vector field $F(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ is an example of why the "no holes" property is needed. F respects the Clairaut Theorem conditions on the region on which it is defined, which is all of $\mathbb{R}^2$ except the origin. But we know it is not conservative on that region, since the line integral of F around the unit circle does not equal zero. The problem is that the region on which F is defined has a hole at the origin.

Kinetic Energy:

Definition: The *kinetic energy* of an object of mass m moving at speed $\frac{ds}{dt}$ is $\frac{m}{2}\left(\frac{ds}{dt}\right)^2$.

Theorem: If F is the total force acting on an object moving along γ, the work done by F is equal to the change in kinetic energy.

Proof: Suppose the object's path is parametrized by $\vec{r}(t)$ for $a \le t \le b$. Then the object's acceleration at time t is $\vec{r}''(t)$. By Newton's Second Law, force equals mass times acceleration, we have

$$F(\vec{r}(t)) = m\vec{r}''(t).$$

Therefore the work done by F is

$$\int_\gamma F \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b m\vec{r}''(t) \cdot \vec{r}'(t)\, dt = \frac{m}{2}\int_a^b 2\vec{r}''(t) \cdot \vec{r}'(t)\, dt.$$

We can check (using the dot product rule) that

$$\frac{d}{dt}\left(\vec{r}'(t) \cdot \vec{r}'(t)\right) = 2\vec{r}''(t) \cdot \vec{r}'(t).$$

Substituting into the equation above gives us

$$\int_\gamma F \cdot d\vec{r} = \frac{m}{2}\int_a^b \frac{d}{dt}\left(\vec{r}'(t) \cdot \vec{r}'(t)\right)\, dt = \frac{m}{2}\left(\vec{r}'(t) \cdot \vec{r}'(t)\right)\Big|_{t=a}^{t=b} = \frac{m}{2}\left|\vec{r}'(t)\right|^2\Big|_{t=a}^{t=b} = \frac{m}{2}\left(\frac{ds}{dt}\right)^2\Big|_{t=a}^{t=b}.$$

But this is precisely the change in kinetic energy.

Theorem: If F is a conservative force $F = \nabla f$ with potential energy function $-f$, and F is the only force acting on an object moving along γ, then the net increase in kinetic energy equals the net decrease in potential energy. That is,

$$\text{kinetic energy} \;+\; \text{potential energy} \;=\; \text{constant}.$$

Proof: By the theorem, since F is the only force acting, the work done by F is the net increase in kinetic energy. By the Fundamental Theorem of Line Integrals, the work done by F is also the net increase in f, which is the net decrease in $-f$, or the net decrease in potential energy.

Theorem: If F is a continuous vector field on an open connected region D, and $\int_\gamma F \cdot d\vec{r}$ is path independent on D, then F is conservative on D.

Proof: We'll do the proof for $\mathbb{R}^2$, but it works for any $\mathbb{R}^n$.

Let (x_0, y_0) be some point in D. We want to find a function f on D with $\nabla f = F$. Since we can add any constant we want without changing the gradient, we can specify that we want $f(x_0, y_0) = 0$.

Suppose (x, y) is another point in D, and γ is a smooth curve in D from (x_0, y_0) to (x, y). From the Fundamental Theorem of Line Integrals, we know that

$$\int_\gamma F \cdot d\vec{r} = \int_\gamma \nabla f \cdot d\vec{r} = f(x, y) - f(x_0, y_0) = f(x, y).$$

This tells us what f must be.

Define a function f on D by

$$f(x, y) = \int_\gamma F \cdot d\vec{r} \text{ where } \gamma \text{ goes from } (x_0, y_0) \text{ to } (x, y) \text{ in } D.$$

Because the line integral gives the same value no matter what path we pick, it makes sense to define a function this way. We need to show that $\nabla f = F$.

Suppose $F(x, y) = \langle P(x, y), Q(x, y) \rangle$. We will show that $f_x(x, y) = P(x, y)$. The proof that $f_y(x, y) = Q(x, y)$ is the same.

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

Let γ be a path from (x_0, y_0) to (x, y), and let ψ be the straight line from (x, y) to $(x + h, y)$. Then $\gamma + \psi$ (this means γ followed by ψ) is a path from (x_0, y_0) to $(x + h, y)$, and we have

$$f(x + h, y) - f(x, y) = \int_{\gamma + \psi} F \cdot d\vec{r} - \int_\gamma F \cdot d\vec{r} = \int_\psi F \cdot d\vec{r}.$$

Now to show
$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = P(x,y)$$
we need to show
$$\lim_{h\to 0}\left(\frac{1}{h}\int_\psi F\cdot d\vec{r}\right) = P(x,y).$$
To compute the line integral in this equation, we parametrize ψ by
$$\vec{r}(t) = \langle x+t, y\rangle \qquad 0\le t\le h \qquad d\vec{r} = \vec{r}\,'(t)\,dt = \langle 1,0\rangle\ dt.$$
Then
$$\frac{1}{h}\int_\psi F\cdot d\vec{r} = \frac{1}{h}\int_0^h \langle P(x+t,y), Q(x+t,y)\rangle\cdot\langle 1,0\rangle\ dt = \frac{1}{h}\int_0^h P(x+t,y)\,dt.$$
If h is very small, then $P(x+t,y)\approx P(x,y)$ for every t between 0 and h, and so we have
$$\frac{1}{h}\int_0^h P(x+t,y)\,dt \approx \frac{1}{h}\int_0^h P(x,y)\,dt = \frac{1}{h}\,(hP(x,y)) = P(x,y).$$
In the limit,
$$\lim_{h\to 0}\left(\frac{1}{h}\int_\psi F\cdot d\vec{r}\right) = P(x,y).$$
This is what we needed to show.

This page is cultural enrichment.

To be more formal about that limit argument:
To show

$$\lim_{h\to 0}\left(\frac{1}{h}\int_{\psi} F\cdot d\vec{r}\right) = P(x,y),$$

we need to show that for every $\varepsilon > 0$, there is a $\delta < 0$, such that whenever $|h| < \delta$ we have

$$\left|\left(\frac{1}{h}\int_{\psi} F\cdot d\vec{r}\right) - P(x,y)\right| < \varepsilon.$$

Because F is continuous, we can choose δ small enough so that whenever $|t| < \delta$ we have $|P(x+t,y) - P(x,y)| < \varepsilon$. That is to say,

$$P(x,y) - \varepsilon < P(x+t,y) < P(x,y) + \varepsilon.$$

Choose δ this small. Then whenever $|h| < \delta$ and t is between 0 and h, so also $|t| < \delta$, we have

$$P(x,y) - \varepsilon < P(x+t,y) < P(x,y) + \varepsilon.$$

This means we have

$$\int_0^h (P(x,y)-\varepsilon)\,dt < \int_0^h P(x+t,y)\,dt < \int_0^h (P(x,y)+\varepsilon)\,dt.$$

But in the integrals on the left and right we are integrating a constant (not a function of t), so we can evaluate those integrals.

$$h(P(x,y)-\varepsilon) < \int_0^h P(x+t,y)\,dt < h(P(x,y)+\varepsilon)$$

$$(P(x,y)-\varepsilon) < \frac{1}{h}\int_0^h P(x+t,y)\,dt < (P(x,y)+\varepsilon)$$

$$(P(x,y)-\varepsilon) < \frac{1}{h}\int_{\psi} F\cdot d\vec{r} < (P(x,y)+\varepsilon)$$

$$\left|\left(\frac{1}{h}\int_{\psi} F\cdot d\vec{r}\right) - P(x,y)\right| < \varepsilon.$$

This is what we needed to show.

Warm-up for next time: Suppose D is a Type I region in the plane, given by $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$, and $P(x, y)$ is a function on D with continuous partial derivatives.

We look at what happens if we integrate $P_y(x, y)$ over D.

$$\iint_D P_y(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} P_y(x,y)\,dy\,dx = \int_a^b (P(x,y))\Big|_{y=g(x)}^{y=h(x)} dx =$$

$$\int_a^b (P(x,h(x)) - P(x,g(x))\,dx = \int_a^b P(x,h(x))\,dx - \int_a^b P(x,g(x))\,dx.$$

Now we look at what happens if we integrate $F(x, y) = \langle P(x, y), 0\rangle$ around the boundary of D, oriented counterclockwise:

We break up the boundary of D into four pieces.

The bottom edge of the boundary γ_1 is parametrized by $\vec{r}(t) = \langle t, g(t)\rangle$ for $a \leq t \leq b$. Then $\vec{r}\,'(t) = \langle 1, g'(t)\rangle$ and

$$\int_{\gamma_1} F \cdot d\vec{r} = \int_a^b \langle P(t,g(t)),0\rangle \cdot \langle 1, g'(t)\rangle\ dt = \int_a^b P(t,g(t))\,dt = \int_a^b P(x,g(x))\,dx.$$

Another way to do the bottom edge:

$$\int_{\gamma_1} F \cdot d\vec{r} = \int_{\gamma_1} \langle P(x,y),0\rangle \cdot \langle dx, dy\rangle = \int_{\gamma_1} P(x,y)\,dx = \int_a^b P(x,g(x))\,dx.$$

Example: In general, if γ and ψ are two different smooth curves starting and ending at the same points, and F is a vector field that is not conservative, then

$$\int_\gamma F \cdot \vec{T}\, ds \neq \int_\psi F \cdot \vec{T}\, ds.$$

Show this by evaluating both integrals when γ is the portion of the unit circle from $(1,0)$ counterclockwise to $(-1,0)$, and ψ is the straight line segment from $(1,0)$ to $(-1,0)$, and $F(x,y) = \langle -y, x \rangle$.

For practice, try evaluating each line integral in three different ways, first by parametrizing the curve and evaluating the line integral as $\displaystyle\int F \cdot d\vec{r}$, second by evaluating separately both pieces of the line integral in the form $\displaystyle\int P\, dx + Q\, dy$, and third by figuring out what $F \cdot \vec{T}$ is on the curve and using that to evaluate the line integral as $\displaystyle\int F \cdot \vec{T}\, ds$.

Example: A force field is given by

$$F(x, y, z) = \langle y, 2x, y \rangle$$

and γ is the intersection of the surfaces $y = x^2$ and $z = x^3$.

Show that F is not conservative.

Find the work done by F on an object that moves along γ from $(0, 0, 0)$ to $(1, 1, 1)$.