Math 11
Fall 2016
Section 1
Monday, October 31, 2016

First, some important points from the last class:

For a continuous vector field F on an open connected region D, we consider the following properties:

(1.) F is conservative (F is a gradient field) on D.

(2.) $\int_\gamma F \cdot d\vec{r}$ is *path independent* on D. This means that if γ and ψ are two oriented curves in D with the same starting and ending points, then $\int_\gamma F \cdot d\vec{r} = \int_\psi F \cdot d\vec{r}$.

(3.) If γ is a smooth closed curve in D (closed means its end point equals its beginning point), then $\int_\gamma F \cdot d\vec{r} = 0$.

(4.) F respects the Clairaut Theorem conditions: If $F = \langle P, Q \rangle$, then $P_y = Q_x$. If $F = \langle P, Q, R \rangle$, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.

(1) $\implies$ (2) by the Fundamental Theorem of Line Integrals.

(2) $\iff$ (3), because if γ and ψ have the same beginning and ending point, then $\gamma + (-\psi)$ (which means γ followed by ψ in the reverse direction) is closed.

(1) $\implies$ (4), provided the components of F have continuous partial derivatives, by Clairaut's Theorem.

(2) $\implies$ (1): Choose a point $\vec{x}_0$ in D, and define $f(\vec{x}) = \int_\gamma F \cdot d\vec{r}$, where γ is any curve in D from $\vec{x}_0$ to $\vec{x}$. This makes sense as a definition because the line integral is path independent. Prove $\nabla f = F$.

(4) $\implies$ (3), provided D is a *simply connected* (no holes) region in $\mathbb{R}^2$ and the components of F have continuous partial derivatives. This will follow from Green's Theorem.

Definition: The *kinetic energy* of an object of mass m moving at speed $\frac{ds}{dt}$ is $\frac{m}{2}\left(\frac{ds}{dt}\right)^2$.

Theorem: If F is the total force acting on an object moving along γ, the work done by F is equal to the change in kinetic energy.

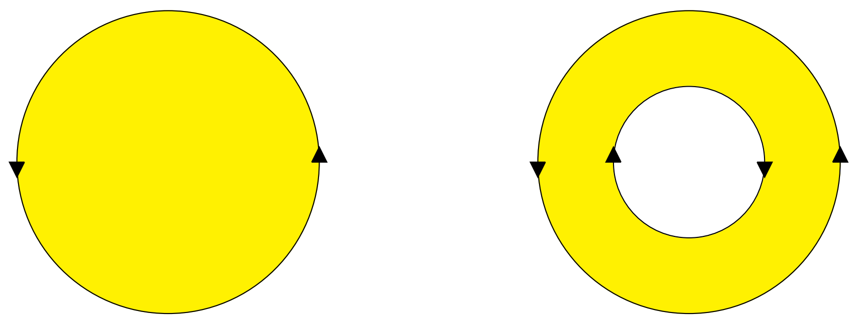
Theorem: If F is a conservative force and F is the only force acting on an object moving along γ, then the net increase in kinetic energy equals the net decrease in potential energy.

$$\text{kinetic energy} \ + \ \text{potential energy} \ = \ \text{constant.}$$

Definition: A curve γ is *closed* if its beginning point equals its end point, *simple* if it does not cross itself, and *piecewise smooth* if it can be divided into finitely many smooth curves.

Definition: If D is a closed, simply connected region in $\mathbb{R}^2$ and γ is the boundary of D, the *positive* orientation on γ circles D counterclockwise.

Another way to say this is that when moving along γ in the direction of the orientation, the inside of the region D is on your left. If D is not simply connected, this gives the correct positive orientation on curves bounding holes in D as well:



Green's Theorem: If $F = \langle P, Q\rangle$ is a vector field on an open region E that includes a closed region D that is both Type I and Type II (or can be divided into finitely many such regions), the components of F have continuous partial derivatives on E, and ∂D is the positively-oriented boundary of D, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

We use Green's Theorem to prove (4) $\implies$ (3), provided D is a *simply connected* (no holes) region in $\mathbb{R}^2$ and the components of F have continuous partial derivatives.

For simplicity, assume γ is a simple closed curve, so we can view γ as the boundary of some region D, perhaps with the negative orientation.

(4) says that $F = \langle P, Q\rangle$ respects the Clairaut Theorem conditions: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

By Green's Theorem, then,

$$\int_\gamma F\cdot d\vec{r} = \pm\int_{\partial D} P\,dx + Q\,dy = \pm\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \pm\iint_D (0)\,dA = 0.$$

There are at least three different ways to use Green's Theorem to evaluate integrals:

Evaluate a line integral around a closed curve by instead evaluating a double integral.

Evaluate a double integral by instead evaluating a line integral around a closed curve.

Evaluate a line integral by instead evaluating a different line integral and a double integral.

Example: Use Green's Theorem to evaluate

$$\int_\gamma F \cdot d\vec{r}$$

where γ is the unit circle with the clockwise orientation, and

$$F(x, y) = \left\langle y^2 - xy + y,\, 2xy \right\rangle.$$

Set $P(x, y) = y^2 - xy + y$ and $Q(x, y) = 2xy$, and D to be the unit disc. The (positively oriented) boundary of D is $-\gamma$. (This means γ with the opposite orientation.) Use Green's Theorem.

$$\int_\gamma F \cdot d\vec{r} = \int_{-\partial D} P\,dx + Q\,dy = -\int_{\partial D} P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$$

$$-\iint_D ((2y) - (2y - x + 1))\; dA = \underbrace{-\iint_D x\,dA}_{\text{0 by symmetry}} + \iint_D 1\,dA = 0 + \text{area}(D) = \pi.$$

Example: Use Green's Theorem to find the area of the region D enclosed by the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1.$$

If we set

$$F(x,y) = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle = \langle P(x,y), Q(x,y) \rangle,$$

then we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1,$$

and so by Green's Theorem, the area of D is

$$\iint_D 1\, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{\partial D} P\, dx + Q\, dy = \int_{\partial D} F \cdot d\vec{r}.$$

We can parametrize the boundary of D:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1.$$

$$(x^{\frac{1}{3}})^2 + (y^{\frac{1}{3}})^2 = 1.$$

$$(x^{\frac{1}{3}}) = \cos t \qquad (y^{\frac{1}{3}}) = \sin t \qquad 0 \le t \le 2\pi.$$

$$x = \cos^3 t \qquad y = \sin^3 t \qquad 0 \le t \le 2\pi.$$

$$\int_{\partial D} F \cdot d\vec{r} = \int_0^{2\pi} \left\langle -\frac{\sin^3 t}{2}, \frac{\cos^3 t}{2} \right\rangle \cdot \left\langle -3(\cos^2 t)(\sin t), 3(\sin^2 t)(\cos t) \right\rangle\, dt =$$

$$\int_0^{2\pi} \frac{3}{2} \left((\cos^2 t)(\sin^4 t) + (\cos^4 t)(\sin^2 t) \right)\, dt = \frac{3}{2} \int_0^{2\pi} (\cos^2 t)(\sin^2 t)(\sin^2 t + \cos^2 t)\, dt =$$

$$\frac{3}{2} \int_0^{2\pi} (\cos^2 t)(\sin^2 t)\, dt = \frac{3}{2} \int_0^{2\pi} \frac{1 + \cos(2t)}{2} \frac{1 - \cos(2t)}{2}\, dt =$$

$$\frac{3}{2} \int_0^{2\pi} \frac{1 - \cos^2(2t)}{4}\, dt = \frac{3}{8} \int_0^{2\pi} 1 - \frac{1 + \cos(4t)}{2}\, dt = \frac{3}{16} \int_0^{2\pi} 1 + \cos(4t)\, dt = \frac{3\pi}{8}.$$

Example: Use Green's Theorem to find

$$\int_\gamma \langle 2xy, x^2 + x\rangle \cdot d\vec{r},$$

where γ is the top half of the unit circle, oriented from left to right.

Let D be the top half of the unit disc, and ψ the portion of the x-axis from -1 to 1. Let $P = 2xy$ and $Q = x^2 + x$. By Green's Theorem,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy.$$

The left-hand side is

$$\iint_D ((2x+1) - 2x)\; dA = \iint_D 1\,dA = \frac{\pi}{2}.$$

The right-hand side is

$$\int_{-\gamma} \vec{F} \cdot d\vec{r} + \int_\psi \vec{F} \cdot d\vec{r}.$$

On ψ we have $y = 0$ and $dy = 0$, so

$$\int_\psi F \cdot d\vec{r} = \int_\psi P\,dx + Q\,dy = \int_\psi P\,dx = \int_{-1}^{1} (2x(0))\,dx = 0.$$

Substituting into our original equation, we have

$$\frac{\pi}{2} = \int_{-\gamma} \vec{F} \cdot d\vec{r} + \int_\psi \vec{F} \cdot d\vec{r} = \int_{-\gamma} \vec{F} \cdot d\vec{r} \;+\; 0 = -\int_\gamma \vec{F} \cdot d\vec{r}.$$

$$\int_\gamma \vec{F} \cdot d\vec{r} = -\frac{\pi}{2}.$$

Proving Green's Theorem:

Use the fact that D is a Type I region to show

$$\iint_D -\frac{\partial P}{\partial y}\, dA = \int_{\partial D} P\, dx.$$

Let D be given by $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$.

$$\iint_D -\frac{\partial P}{\partial y}(x,y)\, dA = -\int_a^b \int_{g(x)}^{h(x)} \frac{\partial P}{\partial y}(x,y)\, dy\, dx = -\int_a^b (P(x,y))\Bigg|_{y=g(x)}^{y=h(x)} dx =$$

$$-\int_a^b (P(x,h(x)) - P(x,g(x))\, dx = -\int_a^b P(x,h(x))\, dx + \int_a^b P(x,g(x))\, dx.$$

We break up ∂D into four pieces.

The bottom edge of the boundary γ_1 is parametrized by $\vec{r}(t) = \langle t, g(t)\rangle$ for $a \leq t \leq b$. Then $\vec{r}\,(t) = \langle 1, g'(t)\rangle$ and

$$\int_{\gamma_1} F \cdot d\vec{r} = \int_a^b \langle P(t,g(t)), 0\rangle \cdot \langle 1, g'(t)\rangle\ dt = \int_a^b P(t,g(t))\, dt = \int_a^b P(x,g(x))\, dx.$$

Another way to do the bottom edge:

$$\int_{\gamma_1} F \cdot d\vec{r} = \int_{\gamma_1} \langle P(x,y), 0\rangle \cdot \langle dx, dy\rangle = \int_{\gamma_1} P(x,y)\, dx = \int_a^b P(x,g(x))\, dx.$$

In the same way, on the top edge γ_2 (which is oriented from right to left, hence the minus sign) we get

$$\int_{\gamma_2} F \cdot d\vec{r} = -\int_a^b P(x,h(x))\, dx.$$

On the left and right edges, we have $dx = 0$.

Putting this together,

$$\int_{\partial D} P\, dx = \int_{\gamma_1} P\, dx + \int_{\gamma_2} P\, dx = \int_a^b P(x,g(x))\, dx - \int_a^b P(x,h(x))\, dx = \iint_D -\frac{\partial P}{\partial y}\, dA.$$

Example: Let

$$F(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

This vector field is not defined at the origin. It satisfies the Clairaut Theorem equations at every point at which it is defined.

Use Green's Theorem to show that if γ is any simple closed curve that circles the origin once counterclockwise, then

$$\int_\gamma F \cdot d\vec{r} = 2\pi.$$

Let ψ be a circle of radius a centered at the origin, oriented counterclockwise, where a is so large that γ lies inside the disc with boundary ψ. Let D be the region between γ and ψ. Then the boundary of D is $\psi + (-\gamma)$. By Green's Theorem,

$$\int_{\partial D} F \cdot d\vec{r} = \iint_D \left(\frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) - \frac{\partial}{\partial y} \left(\frac{-y}{x^2+y^2} \right) \right) = \iint_D 0 \; dA = 0.$$

Now we have

$$0 = \int_{\partial D} F \cdot d\vec{r} = \int_{\psi+(-\gamma)} F \cdot d\vec{r} = \int_\psi F \cdot d\vec{r} = \int_\gamma F \cdot d\vec{r}.$$

$$\int_\gamma F \cdot d\vec{r} = \int_\psi F \cdot d\vec{r}.$$

We can evaluate the second line integral by parametrizing ψ with $x = a\cos t$, $y = a\cos t$, $0 \le t \le 2\pi$. Then

$$\int_\psi \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-a\sin t}{a^2}, \frac{a\cos t}{a^2} \right\rangle \cdot \langle -a\sin t, a\cos t \rangle \; dt = \int_0^{2\pi} \frac{a^2}{a^2} \, dt = 2\pi.$$

Interpreting Green's Theorem:

Consider $F = \langle P, Q \rangle = \langle P, 0 \rangle + \langle 0, Q \rangle$. We will look at the two parts separately.

The first piece $\langle P, 0 \rangle$ is a vector field whose vectors are always parallel to the x-axis. We can think of this as a fluid flow field, with fluid moving parallel to the x-axis.

Suppose $\dfrac{\partial P}{\partial y} > 0$. That means that, at any point, fluid above that point (in the positive y-direction) is moving to the right faster than fluid below that point. (This supposes $P > 0$. If $P < 0$, fluid above that point is moving to the left slower than fluid below that point.)

Now imagine a small paddle wheel fixed to the xy-plane, and free to turn clockwise or counterclockwise. Since fluid on the top part of the wheel is moving to the right faster than fluid on the bottom part, the net effect is a tendency for the top part of the wheel to move right — that is, a tendency for the wheel to turn clockwise.

By the same reasoning, if $\dfrac{\partial Q}{\partial x} > 0$, the flow field $\langle 0, Q \rangle$ produces a tendency for the wheel to turn counterclockwise.

Putting this together, $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is the counterclockwise rotational tendency produced by the vector field F. If $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} > 0$ on D, then overall F produces a counterclockwise rotational tendency. By Green's Theorem, we also have $\displaystyle\int_{\partial D} F \cdot d\vec{r} > 0$, meaning that overall F acts in the direction of counterclockwise motion around the boundary of D. This makes intuitive sense.

The integral $\displaystyle\int_{\partial D} F \cdot d\vec{r}$ is sometimes called the *circulation* of F around ∂D. Green's Theorem says that the circulation of F around the boundary of D is the same as the integral of the counterclockwise rotational tendency of F over D.

There is a second interpretation of Green's Theorem, which we will see later. There are two different three-dimensional analogues of Green's Theorem, associated with these two different interpretations.

The analogue associated with this interpretation is Stokes' Theorem, which says that if S is a nice surface in $\mathbb{R}^3$ whose boundary is a closed curve γ in $\mathbb{R}^3$, and F is a vector field in $\mathbb{R}^3$, then $\displaystyle\int_{\gamma} F \cdot d\vec{r}$ equals the integral over S of the rotational tendency produced by F. This rotational tendency is actually a vector (indicating the direction of the axis of rotation as well as the magnitude of the rotational tendency). We are about to learn how this rotational tendency is defined, and how to integrate over a surface in three dimensions.

Example: Use Green's Theorem to find the area of the quadrilateral with vertices $(2,1)$, $(-1,3)$, $(2,-1)$, and $(-2,-2)$. Note that there are three natural choices for $\langle P, Q\rangle$ that give $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$: $\langle -y, 0\rangle$, $\langle 0, x\rangle$, and $\left\langle \dfrac{-y}{2}, \dfrac{x}{2}\right\rangle$.

Example: Verify Green's Theorem for the vector field

$$F(x,y) = \langle x - y, x + y \rangle$$

and the region D defined by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1.$$

(Green's Theorem states that two integrals are equal. To verify it in this case means to evaluate each integral directly, and check that you get the same value for both.)