Math 11
Fall 2016
Section 1
Wednesday, November 2, 2016

First, some important points from the last class:

Definition: A curve γ is *closed* if its beginning point equals its end point, *simple* if it does not cross itself, and *piecewise smooth* if it can be divided into finitely many smooth curves.

Definition: If D is a closed, simply connected region in $\mathbb{R}^2$ and γ is the boundary of D, the *positive* orientation on γ circles D counterclockwise.

Another way to say this is that when moving along γ in the direction of the orientation, the inside of the region D is on your left. If D is not simply connected, this gives the correct positive orientation on curves bounding holes in D as well.

Green's Theorem: If $F = \langle P, Q \rangle$ is a vector field on an open region E that includes a closed region D that is both Type I and Type II (or can be divided into finitely many such regions), the components of F have continuous partial derivatives on E, and ∂D is the positively-oriented boundary of D, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

There are at least three different ways to use Green's Theorem to evaluate integrals:

Evaluate a line integral around a closed curve by instead evaluating a double integral.

Evaluate a double integral by instead evaluating a line integral around a closed curve.

Evaluate a line integral by instead evaluating a different line integral and a double integral.

$\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ is the (counterclockwise) rotational tendency produced by the vector field F.

The integral $\displaystyle\int_{\partial D} F \cdot d\vec{r}$ is sometimes called the *circulation* of F around ∂D.

Green's Theorem says that the circulation of F around the boundary of D equals the integral over D of the rotational tendency of F.

Today: Curl (rotational tendency) and divergence (expansionary tendency) of vector fields.

Notation: We sometimes write

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

∇, called "del," is a vector differential operator. In this context, an operator is a kind of function, for which the inputs and outputs are themselves functions. $\dfrac{d}{dx}$ and $\dfrac{\partial}{\partial x}$ are examples of scalar differential operators.

We use ∇ to keep notational track of various combinations of partial derivatives we have defined or are about to define. For example, if f is a function from $\mathbb{R}^3$ to $\mathbb{R}$, then the gradient of f is

$$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

We may use the same notation in other dimensions. However, the following definition is only for $\mathbb{R}^3$. (You can see that, because it involves the cross product.)

Definition: If $F = \langle P, Q, R \rangle$ is a three-dimensional vector field, then the *curl* of F is

$$\operatorname{curl}(F) = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

Notice that the curl of F is also a three-dimensional vector field. It represents the rotational tendency of F (which is generally different at different points). The direction of the curl is the axis around which rotation tends to occur (counterclockwise as viewed from the arrow end of the vector), and the magnitude is the strength of the rotational tendency.

Example: If $F = \langle P(x,y), Q(x,y), 0\rangle$, then F is essentially a two-dimensional vector field $\langle P(x,y), Q(x,y)\rangle$ in the xy plane, lifted into three dimensions. Since P and Q are functions of x and y, their partial derivatives with respect to z are zero. The curl of F is

$$\text{curl}(F) = \nabla \times \langle P, Q, 0\rangle = \left\langle 0,\ 0,\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle .$$

The direction of the curl is either $\mathbf{k}$ or $-\mathbf{k}$, so the rotational tendency of F is rotation around a vertical axis, or in a horizontal plane. The magnitude of the curl, the strength of the rotational tendency, is the absolute value of $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$. We saw last time that this is the rotational tendency of the two-dimensional vector field $\langle P(x,y), Q(x,y)\rangle$.

Example: If $F(x,y,z) = \langle z^2, y^2, x^2 \rangle$, what is the curl of F?

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y^2 & x^2 \end{vmatrix} = \langle 0, 2z - 2x, 0 \rangle \,.$$

Theorem: If $f : \mathbb{R}^3 \to \mathbb{R}$ has continuous second partial derivatives, then $\nabla \times (\nabla f) = \vec{0}$.

Another way to say this:

If F is a conservative three-dimensional vector field whose components have continuous partial derivatives, then $\mathrm{curl}(F) = \vec{0}$.

Proof: This follows from Clairaut's Theorem: The components of $\nabla \times (\nabla f)$ are differences of corresponding mixed second partials of f.

Theorem: If F is a three-dimensional vector field on a simply connected open region, the components of F have continuous partial derivatives, and $\mathrm{curl}(F) = \vec{0}$, then F is conservative.

Proof: We proved the two-dimensional version of this using Green's Theorem. This version follows from Stokes' Theorem, which is a three-dimensional version of Green's Theorem, in the same way. We will see Stokes' Theorem later.

Example: Determine whether $F(x, y, z) = \langle y, x, z^2 \rangle$ is conservative. If it is, find a potential function.

$$\operatorname{curl}(F) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z^2 \end{vmatrix} = \langle 0, 0, 0 \rangle .$$

Therefore F is conservative.

To find a potential function:
$f_x = y$
$f = xy + g(y, z)$
$f_y = x + g_y(y, z) = x$
$g_y(y, z) = 0$
$g(y, z) = h(z)$
$f = xy + h(z)$
$f_z = h'(z) = z^2$
$h(z) = \dfrac{z^3}{3} + C$
$f = xy + \dfrac{z^3}{3} + C$
We can take $C = 0$. A potential function is $-f = -xy - \dfrac{z^3}{3}$.

Definition: A vector field whose curl is $\vec{0}$ is called *irrotational*.

It makes physical sense that a conservative vector field should be irrotational. For example, a gravitational field is conservative; it has a potential energy function, and the force acts in the direction from high potential to low potential. Imagine an object — like a paddle wheel — fixed in one place but free to spin. For gravity to impart a rotational tendency would be bizarre.

Definition: The divergence of a vector field F is

$$\text{divergence}(F) = \nabla \cdot F.$$

$$\nabla \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Note that if F is a vector field, its curl $\nabla \times F$ is a vector function, but its divergence $\nabla \cdot f$ is a scalar function.

Theorem: If $F = \langle P, Q, R \rangle$ and the components of F have continuous second partial derivatives, then $\nabla \cdot (\nabla \times F) = 0$.

Proof: This follows from Clairaut's Theorem.

Example: Is the vector field $F(x, y, z) = \langle -y, x, z^2 \rangle$ the curl of some other vector field?

Check its divergence:

$$\nabla \cdot F = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial(z^2)}{\partial z} = 2z.$$

Because the divergence is not 0, we cannot write F as a curl.

Note: We are not claiming, in this course, that if the divergence of F is zero then F can be written as a curl. However, under appropriate assumptions (including that the region in question is simply connected), this is true.

If we view F as a fluid flow field, the divergence $\nabla \cdot F$ gives the expansionary tendency of the flow.

Definition: If $\nabla \cdot F = 0$, then F is called *incompressible*.

Note: Provided everything in sight is sufficiently differentiable, Clairaut's Theorem tells us that

$$\nabla \times (\nabla f) = \vec{0}$$

$$\nabla \cdot (\nabla \times F) = 0.$$

However, this does not mean we always get zero by applying ∇ twice. It is quite possible to have, for example,

$$\nabla \times (\nabla \times F) \neq \vec{0}$$

$$\nabla \cdot (\nabla f) \neq 0,$$

while, for example,

$$\nabla \cdot (\nabla \cdot F)$$

doesn't even make sense.

Example: If $f(x, y, z) = x^2 + y^2 + z^2$ then

$$\nabla \cdot (\nabla f) = \nabla \cdot \langle 2x, 2y, 2z \rangle = 2 + 2 + 2 = 6.$$

Definition: We write

$$\nabla^2 = \nabla \cdot \nabla,$$

so $\nabla^2 f = \nabla \cdot (\nabla f)$. The partial differential equation

$$\nabla^2 f = 0$$

is called Laplace's equation, and ∇^2 is called the Laplace operator.

Laplace's equation is important in physics. For example, if $f(x, y, z)$ denotes the temperature at a point (x, y, z), and f satisfies Laplace's equation, then this is a steady state temperature distribution.

Heat flows from regions of high temperature to regions of low temperature; the heat flow is (some constant multiple of) $-\nabla f$. If the divergence of this flow is zero, that means that heat flowing into a region equals heat flowing out, and temperature remains unchanged.

Vector Forms of Green's Theorem:

Suppose that D is a region in the xy plane, P and Q are functions of x and y, and D, P, and Q satisfy Green's Theorem. We can view the xy-plane as sitting inside $\mathbb{R}^3$, and write $F = \langle P, Q, 0\rangle$. Then we have

$$\operatorname{curl}(F) = \nabla \times F = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$\int_{\partial D} F \cdot T\, ds = \int_{\partial D} P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D \operatorname{curl}(F) \cdot \mathbf{k}\, dA.$$

That is, the line integral of the tangential component of F around the boundary of D equals the integral of the vertical component of the curl of F over D.

(The vertical component is normal to the horizontal plane containing D. The component of a vector function normal to a surface, integrated over that surface, will show up again later.)

We can also integrate the normal component of F around the boundary of D: Let $\vec{n}$ be the unit vector normal to ∂D and pointing outward from D. Then we have:

$$\vec{T}\, ds = \langle dx, dy \rangle \qquad \vec{n}\, ds = \langle dy, -dx \rangle .$$

$$\int_{\partial D} F \cdot \vec{n}\, ds = \int_{\partial D} \langle P, Q \rangle \cdot \langle dy, -dx \rangle = \int_{\partial D} (-Q)\, dx + P\, dy.$$

Applying Green's theorem to $\langle -Q, P\rangle$, this is

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_D \nabla \cdot F\, dA.$$

That is, the line integral of the normal component of F around the boundary of D equals the integral of the divergence of F over D.

If F is a fluid flow field, then the line integral of the normal component F around the boundary of D represents the net rate of flow out of the region D.

The picture below illustrates this.

The horizontal green arrow represents a small displacement along the oriented curve ∂D of length ds.

The vertical black arrow represents a normal vector $\vec{n}\,ds$ of length ds.

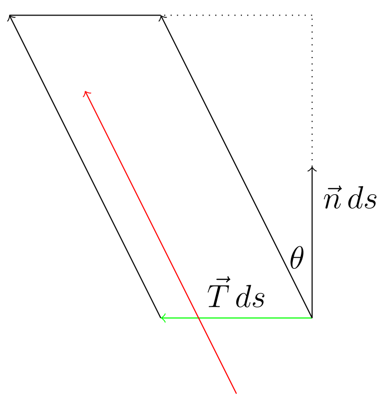
The slanted red arrow represents F, as do the slanted sides of the parallelogram. The normal component of F is $F \cdot \vec{n} = |F| \cos\theta$, or the height of the illustrated parallelogram.

If F is the velocity of fluid flow, then a particle of fluid moves in one unit of time from the beginning to the end of the red arrow representing F. Notice that it crosses the green arrow, and ends up inside the parallelogram.

The fluid inside the parallelogram is all the fluid that crossed the green arrow during the last unit of time. The area of the parallelogram, which is

$$(ds)(\text{height}) = F \cdot \vec{n}\,ds,$$

is the amount of fluid that flows over that green arrow during one unit of time. That is, it is the rate of flow across the green portion of ∂D.



To find the rate of flow across all of ∂D, we add up the rates of flow across all its infinitesimal pieces, by taking the integral

$$\int_{\partial D} F \cdot n\,ds.$$

If this is positive, then the net flow across the boundary of D is outward; if it is negative, the net flow is inward.

A region on which $\nabla \cdot F > 0$ is a fluid source, and one on which $\nabla \cdot F < 0$ is a fluid sink.

Example: Let $F(x, y) = \langle x, y\rangle$ and D be the unit disc. Then ∂D is the counterclockwise unit circle, which we can parametrize by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

$$\iint_D \nabla \cdot F\, dA = \iint_D \left(\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y}\right)\, dA = \iint_D 2\, dA = 2\pi.$$

$$\int_{\partial D} F \cdot \vec{n}\, ds = \int_{\partial D} \langle x, y\rangle \cdot \langle dy, -dx\rangle = \int_{\partial D} -x\, dy + y\, dx =$$

$$\int_0^{2\pi} -\cos t(-\cos t\, dt)\ +\ \sin t(\sin t\, dt) = \int_0^{2\pi} 1\, dt = 2\pi.$$

Example: Let D be the same region, and

$$G(x, y) = \left\langle \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right\rangle.$$

Then

$$\int_{\partial D} G \cdot \vec{n}\, ds = \int_{\partial D} \left\langle \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right\rangle \cdot \langle dy, -dx\rangle = \int_{\partial D} -\frac{x}{x^2 + y^2}\, dy + \frac{y}{x^2 + y^2}\, dx =$$

$$\int_0^{2\pi} -\cos t(-\cos t\, dt)\ +\ \sin t(\sin t\, dt) = \int_0^{2\pi} 1\, dt = 2\pi.$$

We can check that $\nabla \cdot G = 0$ at every point except the origin, at which G is not defined.

We can picture these examples by imagining the unit disc as the top of a round table over which a thin film of water is flowing, flowing out over the unit circle at the edge at a rate of 2π units per second. In both cases, the direction of flow is directly away from the origin.

In the first case, the divergence of F has a constant value of 2. This means that the source of fluid is evenly distributed over the table top. Perhaps it is raining.

In the second case, the divergence of G is zero, except at the origin, where there is a discontinuity. All the fluid must be originating from the origin. It looks like we have a fountain here.

You can check that if γ is any circle around the origin, the rate of flow of F across γ equals twice the area enclosed by γ, and the rate of flow of G across γ equals 2π.

Example: Find the divergence and the curl of the vector function

$$F(x, y, z) = \langle x \sin y, x \cos y, xyz \rangle .$$

Is F conservative? Can F be written as the curl of some vector field?

Example: If $f : \mathbb{R}^3 \to \mathbb{R}$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$, identify each of the following expressions as representing a vector field, representing a scalar field, or meaningless.

Which of these, provided everything is sufficiently differentiable, are necessarily zero?

div(grad(f))	div(grad(F))
div(curl(f))	div(curl(F))
div(div(f))	div(div(F))
grad(grad(f))	grad(grad(F))
grad(curl(f))	grad(curl(F))
grad(div(f))	grad(div(F))
curl(grad(f))	curl(grad(F))
curl(curl(f))	curl(curl(F))
curl(div(f))	curl(div(F))

Example: Suppose $F(x,y) = \langle x^2, y^2 \rangle$ and D is the region in $\mathbb{R}^2$ bounded by the curves $y = x^2$ and $y = 1$. Find

$$\int_{\partial D} F \cdot \vec{n}\, ds$$

in two ways, by directly evaluating the line integral and by using Green's Theorem.