Math 11
Fall 2016
Section 1
Friday, November 4, 2016

First, some important points from the last class:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

$$\mathrm{grad}(f) = \nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

$$\mathrm{curl}(F) = \nabla \times F$$

$$\nabla \times \langle P, Q, R \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

$$\mathrm{div}(F) = \nabla \cdot F.$$

$$\nabla \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Theorem: If $f : \mathbb{R}^3 \to \mathbb{R}$ has continuous second partial derivatives, then $\nabla \times (\nabla f) = \vec{0}$. In other words, if F is a conservative three-dimensional vector field whose components have continuous partial derivatives, then $\mathrm{curl}(F) = \vec{0}$.

Theorem: If F is a three-dimensional vector field on a simply connected open region, the components of F have continuous partial derivatives, and $\mathrm{curl}(F) = \vec{0}$, then F is conservative.

Theorem: If $F = \langle P, Q, R \rangle$ and the components of F have continuous second partial derivatives, then $\nabla \cdot (\nabla \times F) = 0$.

Definition: A vector field whose curl is $\vec{0}$ is called *irrotational*. A vector field whose divergence is 0 is called *incompressible*.

Definition: The Laplace operator is $\nabla^2 = \nabla \cdot \nabla$, and $\nabla \cdot (\nabla f) = 0$ is Laplace's equation.

Green's Theorem in vector forms: ($\vec{T}\, ds = \langle dx, dy \rangle$ and $\vec{n}\, ds = \langle dy, -dx \rangle$).

$$\int_{\partial D} F \cdot T\, ds = \iint_D \mathrm{curl}(F) \cdot \mathbf{k}\, dA.$$

$$\int_{\partial D} F \cdot \vec{n}\, ds = \iint_D \mathrm{div}(F)\, dA.$$

Today: Parametrizing surfaces, and computing surface area.

Our first example of a parametrized curve came from the vector parametric equation for a line,

$$\vec{r} = \vec{r}_0 + t\vec{v},$$

where t is a real number parameter. Each choice of t gives one point on the line, and each point on the line comes from a choice of t. We then noted that this expresses the line as the range of a vector function

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}.$$

We parametrize other curves by representing them as ranges of other vector functions.

When we were talking about lines and planes, we briefly mentioned that there is a vector parametric equation for a plane: If $\vec{r}_0$ is a point on the plane, and $\vec{p}$ and $\vec{q}$ are two vectors parallel to the plane (but not parallel to each other), then we can write an arbitrary point on the plane as

$$\vec{r} = \vec{r}_0 + u\vec{p} + v\vec{q}$$

where u and v are two real number parameters.. Each choice of u and v gives one point on the plane, and each point on the plane comes from a choice of u and v. This represents the plane as the range of a function $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$

$$\vec{r}(u, v) = \vec{r}_0 + u\vec{p} + v\vec{q}.$$

Parametrize other surfaces by representing them as ranges of other functions from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Example: Parametrize the cylinder $x^2 + y^2 = 1$ by representing it as the range of a function $\vec{r}(u, v)$.

In cylindrical coordinates, the cylinder is given by $r = 1$, so points on the cylinder are already represented in terms of two variables, θ and z. If we set $\theta = u$ and $z = v$, we can represent the cylinder by

$$\langle x, y, z\rangle = \vec{r}(u, v) = \langle \cos u, \sin u, v\rangle \qquad 0 \le u \le 2\pi \qquad -\infty \le v \le \infty.$$

We can parametrize a portion of the cylinder by, for example, taking $0 \le v \le 1$.

Example: Parametrize the portion of the plane $x + 2y - z = 1$ inside the cylinder $x^2 + y^2 = 1$.

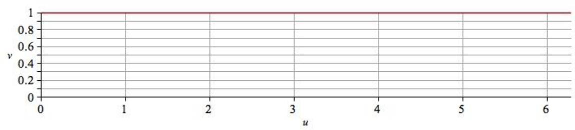
We can rewrite the equation of the plane as

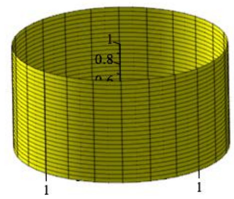
$$z = x + 2y - 1$$

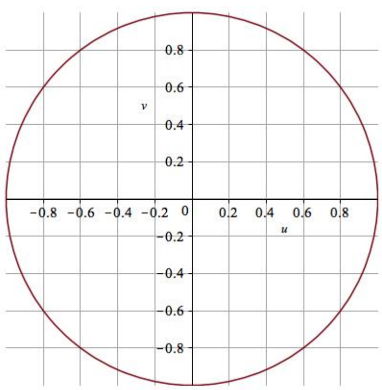
and use rectangular coordinates, setting $x = u$ and $y = v$, to get

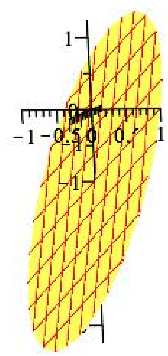
$$\vec{r}(u, v) = \langle u, v, u + 2v - 1\rangle \qquad u^2 + v^2 \le 1.$$

Here are pictures of the domains and ranges of these parametrizations. The lines in the domain are lines $u =$ constant and $v =$ constant. The images of those lines in the range are called *grid curves*. They are also described by $u =$ constant and $v =$ constant.









Example: Parametrize the portion of the cone $z^2 = x^2 + y^2$ for which $0 \leq z \leq 1$.

We can express the cone in rectangular, cylindrical, or spherical coordinates,

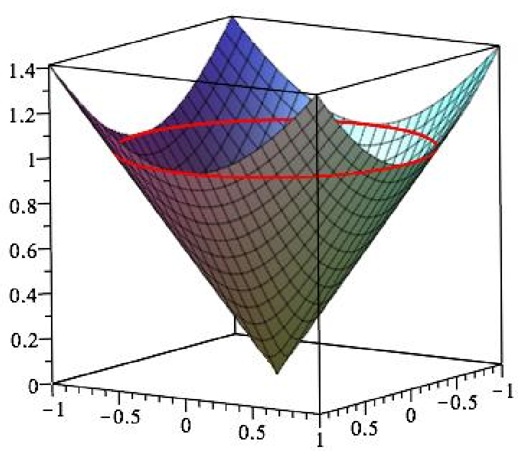
$$z = \sqrt{x^2 + y^2} \qquad z = r \qquad \phi = \frac{\pi}{4}.$$

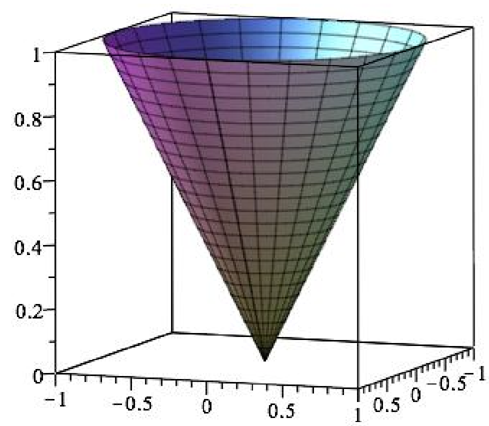
Each representation gives a parametrization:

$$\vec{r}(u, v) = \left\langle u, v, \sqrt{u^2 + v^2} \right\rangle \qquad u^2 + v^2 \leq 1;$$

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, u \rangle \qquad 0 \leq v \leq 2\pi \qquad 0 \leq u \leq 1;$$

$$\vec{r}(u, v) = \left\langle u \cos v \sin\left(\frac{\pi}{4}\right),\ u \sin v \sin\left(\frac{\pi}{4}\right),\ u \cos\left(\frac{\pi}{4}\right) \right\rangle \qquad 0 \leq v \leq 2\pi \qquad 0 \leq u \leq \sqrt{2}.$$





The first picture shows grid curves from the parametrization using rectangular coordinates. The part of the cone we parametrized is the part below the red circle.

The grid curves from the other two parametrizations both look like the ones in the second picture.

Example: Describe the surface parametrized by the function

$$\vec{r}(u, v) = \langle \cos(u + v), v, \sin(u + v) \rangle \, ,$$

and describe the grid curves.

We have $x = \cos(u + v)$ and $z = \sin(u + v)$, so $x^2 + z^2 = \cos^2(u + v) + \sin^2(u + v) = 1$. The surface is a portion of the cylinder $x^2 + z^2 = 1$, a cylinder of radius 1 around the y-axis.

To determine whether every point on that cylinder is on this surface, note that every point on the cylinder can be represented as $\langle \cos\theta, y, \sin\theta \rangle$. So the question is whether we can always choose u and v so that $y = v$, $\cos\theta = \cos(u + v)$, and $\sin\theta = \sin(u + v)$. The answer is yes; choose $v = y$ and $u = \theta - y$. So this function parametrizes that entire cylinder.

The grid lines for $u = a$, where a is some constant, have the form

$$x = \cos(a + v) \qquad y = v \qquad z = \sin(a + v).$$

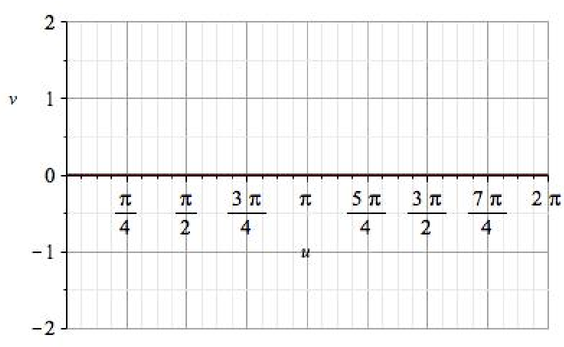
This is a helix; as v increases, $y = v$ increases, while $\langle x, z \rangle = \langle \cos(a + v), \cos(a + v) \rangle$ moves around the unit circle in the xz plane.

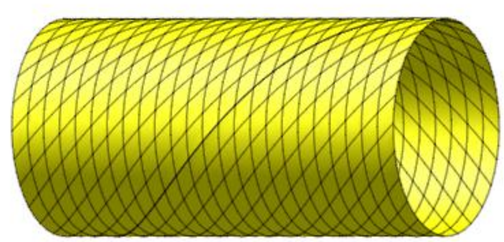
The grid lines for $v = b$, where b is some constant, have the form

$$x = \cos(u + b) \qquad y = b \qquad z = \sin(u + b).$$

This is a circle in the plane $y = b$.

Note that to get the entire cylinder we can restrict to $0 \le u \le 2\pi$.

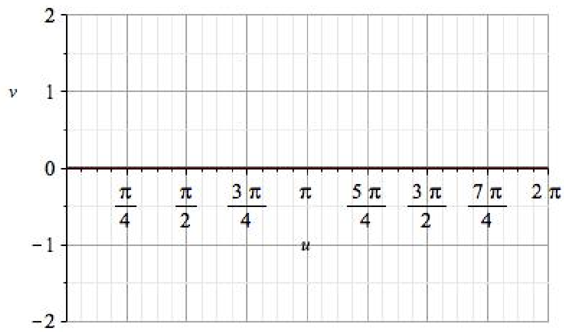


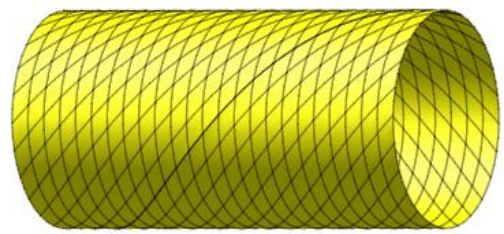


Two things we can do with a parametrization of a surface S:

Find a normal vector to S at a point.

Find the surface area of S.





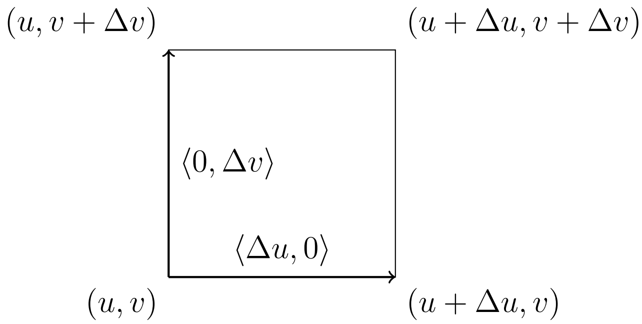
Notice that little rectangles in the uv plane are mapped to little pieces of the surface S that are more or less parallelograms. We will use partial derivatives to approximate the edges of these approximate parallelograms. The cross product of the edges of a parallelogram gives both the area of the parallelogram and a direction normal to the parallelogram.

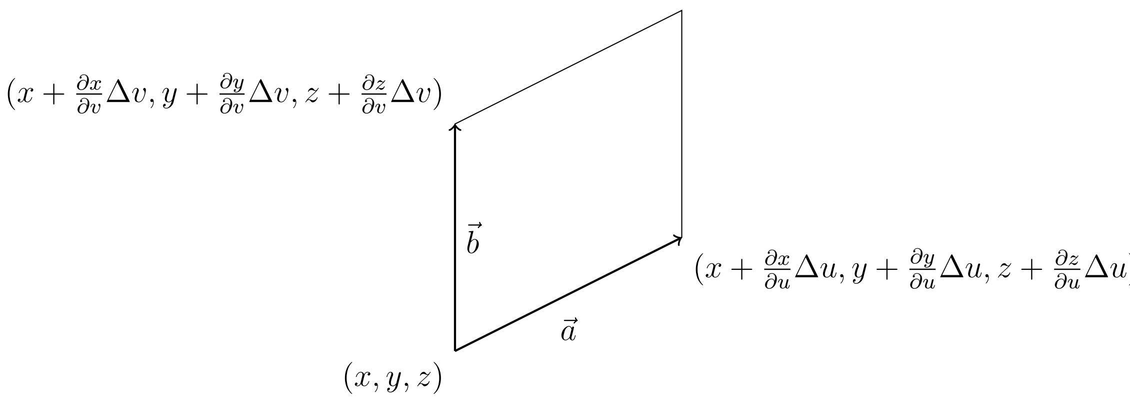
We approximate the surface area of S by adding up the surface areas of all the little parallelograms. As we take smaller and smaller pieces our approximation becomes better and better, and in the limit, we get an integral that gives us the surface area of S.

Suppose our parametrization is

$$\vec{r} = \langle x, y, z \rangle \qquad \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle .$$

Look at a small rectangle in the uv plane of dimensions $\Delta u \times \Delta v$, and its image on the surface S. As u increases by Δu and v remains constant, x increases by approximately $\dfrac{\partial x}{\partial u}\Delta u$, and similarly for y and z.





The image parallelogram has edges

$$\vec{a} = \left\langle \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, \frac{\partial z}{\partial u}\Delta u \right\rangle = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \Delta u = \vec{r}_u\, \Delta u;$$

$$\vec{b} = \left\langle \frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, \frac{\partial z}{\partial v}\Delta v \right\rangle = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \Delta v = \vec{r}_v\, \Delta v.$$

Their cross product is $(\vec{r}_u \times \vec{r}_v)\, \Delta u\, \Delta v$. The vector $\vec{r}_u \times \vec{r}_v$ is normal to the surface, and the element of surface area is

$$dS = \left|\vec{r}_u \times \vec{r}_v\right|\, du\, dv.$$

To find the surface area of S we convert the surface integral $\displaystyle\iint_S dS$ into a double integral over the domain of the parametrization in the uv plane.

Example: Suppose S is the cone $z = \sqrt{x^2 + y^2}$, oriented with normal vector pointing away from the z-axis. Find the tangent plane to S at the point $(3, 4, 5)$, the unit normal vector to S at the same point, and the surface area of the portion of S for which $0 \leq z \leq 1$.

We will use the parametrization from cylindrical coordinates,

$$\vec{r} = \langle x, y, z \rangle = \langle u \cos v, u \sin v, u \rangle \qquad \vec{r}_u = \langle \cos v, \sin v, 1 \rangle \qquad \vec{r}_v = \langle -u \sin v, u \cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -u \cos v, -u \sin v, u \rangle \qquad |\vec{r}_u \times \vec{r}_v| = u\sqrt{2}.$$

To evaluate these at the point $(3, 4, 5)$, we can try to find u_0 and v_0 for which $\vec{r}(u_0, v_0) = (3, 4, 5)$. Or, we can proceed this way: At this point, we have

$$x = u_0 \cos v_0 = 3 \qquad y = u_0 \sin v_0 = 4 \qquad z = u_0 = 5$$

$$\vec{r}_u \times \vec{r}_v = \langle -u_0 \cos v_0, -u_0 \sin v_0, u_0 \rangle = \langle -3, -4, 5 \rangle .$$

This vector is normal to the surface, and therefore to the tangent plane, and a point on the tangent plane is $(3, 4, 5)$, so the equation of the tangent plane is

$$-3(x - 3) - 4(y - 4) + 5(z - 5) = 0.$$

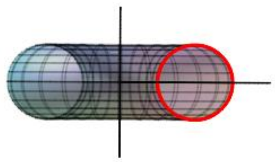
The vector $\langle -3, -4, 5 \rangle$ is normal to S at the point $(3, 4, 5)$. To use it to compute the unit normal vector, we first have to check the orientation. This vector is angled upward and inward; it points toward the z-axis rather than away from it. So we should take the opposite vector, and normalize it (multiply by a positive scalar to get a unit vector):

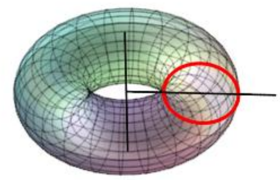
$$\vec{n} = \frac{1}{|\langle 3, 4, -5 \rangle|} \langle 3, 4, -5 \rangle = \frac{1}{5\sqrt{2}} \langle 3, 4, -5 \rangle .$$

To find the surface area of the portion of the cone parametrized by $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$, we use $dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv$:

$$\iint_S dS = \int_0^{2\pi} \int_0^1 |\vec{r}_u \times \vec{r}_v| \, du \, dv = \int_0^{2\pi} \int_0^1 u\sqrt{2} \, du \, dv = \pi\sqrt{2}.$$

Example: The circle in the xz-plane of radius a around the point $x = b$, $z = 0$ (where $a \leq b$) is revolved around the z-axis to form a surface S. (This surface is called a torus.) Parametrize S and find its surface area.





First, we can parametrize the circle in the xz-plane by

$$x = b + a\cos t \qquad z = a\sin t \qquad 0 \leq t \leq 2\pi.$$

Now, regard the half of the xz plane where $x \geq 0$ as a slice of $\mathbb{R}^3$ for fixed θ. The distance x from the z-axis is the r of cylindrical coordinates, so we can replace x with r to get

$$r = b + a\cos t \qquad z = a\sin t \qquad 0 \leq t \leq 2\pi.$$

This will be the case for the slice of the surface S at any θ.

We can parametrize S by setting $u = t$ and $v = \theta$, so

$$x = r\cos\theta = (b + a\cos u)\cos v \qquad y = r\sin\theta = (b + a\cos u)\sin v \qquad z = a\sin u$$

$$0 \leq u \leq 2\pi \qquad 0 \leq v \leq 2\pi.$$

To set up the surface area integral, we have

$$\vec{r}_u = \langle -a\sin u\cos v, -a\sin u\sin v, a\cos u\rangle \qquad \vec{r}_v = \langle -(b + a\cos u)\sin v, (b + a\cos u)\cos v, 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -a(b + a\cos u)\cos u\cos v, -a(b + a\cos u)\cos u\sin v, -a(b + a\cos u)\sin u\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = a(b + a\cos u)$$

$$\iint_S dS = \int_0^{2\pi}\int_0^{2\pi} |\vec{r}_u \times \vec{r}_v|\, du\, dv = \iint_S dS = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos u)\, du\, dv = 4\pi ab.$$

Example: Use the method of the previous example to parametrize the cone obtained by revolving the line segment from $(0,0,0)$ to $(a,0,h)$ around the z-axis, and to find its surface area.

Example: Find the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$ in three ways:

1. by parametrizing the surface with a function $\vec{r}$ and using $\vec{r}_u \times \vec{r}_v$ as the normal vector;
2. by writing the surface as a level surface of a function g and using ∇g as the normal vector;
3. by writing the surface as the graph of a function f and using the tangent plane approximation to f.

Example: Describe and/or sketch the surface parametrized by

$$\vec{r} = \langle u \cos v, u \sin v, v \rangle \qquad 0 \leq u \leq 1 \qquad 0 \leq v \leq 4\pi.$$

Describe and/or sketch its grid curves. Find the tangent plane at the point $(0, .5, .5\pi)$. Find the surface area.