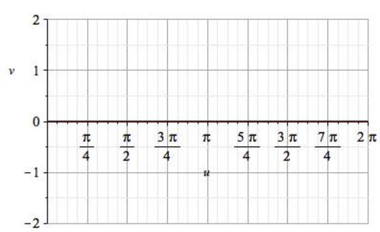
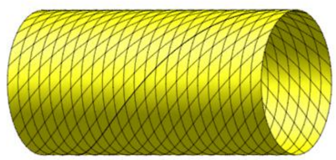
Math 11
Fall 2016
Section 1
Monday, November 7, 2016

First, some important points from the last class:

Parametrize a surface S in $\mathbb{R}^3$ by representing it as the range of a function $\vec{r}(u, v)$.

Lines $u =$ constant and $v =$ constant on the surface are *grid curves.*





If $\vec{r}(u, v) = \langle x, y, z \rangle$ (where x, y, and z are functions of u and v), then:

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \qquad \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

The vector $\vec{r}_u \times \vec{r}_v$ is normal to the surface, and the element of surface area is

$$dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv.$$

To find the surface area of S we convert the surface integral $\iint_S dS$ into a double integral over the domain of the parametrization in the uv plane.

The unit normal vector to S is

$$\vec{n} = \frac{1}{|\vec{r}_u \times \vec{r}_v|} \left(\vec{r}_u \times \vec{r}_v \right).$$

The direction of $\vec{n}$ gives an *orientation* to S. We can think of the side of the surface from which $\vec{n}$ points away as the right side of the surface, and the other as the wrong side.

Today: Surface integrals.

Preview: We had two vector versions of Green's Theorem. If $F = \langle P, Q, 0 \rangle$, where P and Q are functions of x and y, and D is a sufficiently nice region in the xy plan, then we can write Green's Theorem as:

$$\iint_D (\nabla \times F) \cdot \mathbf{k}\, dA = \int_{\partial D} F \cdot T\, ds.$$

That is, the line integral of the tangential component of F around the boundary of D equals the integral of the vertical component of the curl of F over D.

Let $\vec{n}$ be the unit vector normal to ∂D and pointing outward from D in the xy plane. Then we can write Green's Theorem as:

$$\iint_D \nabla \cdot F\, dA = \int_{\partial D} F \cdot \vec{n}\, ds.$$

That is, the line integral of the normal component of F around the boundary of D equals the integral of the divergence of F over D.

Each of these versions of Green's Theorem has a three-dimensional version.

Stokes' Theorem: If S is a sufficiently nice oriented surface in $\mathbb{R}^3$ with positively oriented boundary ∂S, and F is a sufficiently nice vector field, then

$$\iint_S (\nabla \times F) \cdot \vec{n}\, dS = \int_{\partial S} F \cdot T\, ds.$$

The Divergence Theorem: If D is a sufficiently nice three-dimensional region in $\mathbb{R}^3$ with positively oriented boundary ∂D, and F is a sufficiently nice vector field, then

$$\iiint_D (\nabla \cdot F)\, dV = \iint_{\partial D} F \cdot \vec{n}\, dS.$$

Before we can really state these theorems, we need to know what those *surface integrals* $\iint_S (\nabla \times F) \cdot \vec{n}\, dS$ and $\iint_{\partial D} F \cdot \vec{n}\, dS$ are.

First, the integral over the surface S of a scalar function f.

If f is constant with value C, the value of this integral is $(C)(\text{area}(S))$. If f is not constant, we approximate the integral by dividing S into many little nearly parallelogram shaped pieces, multiplying the area of each piece by the value of f at a point on that piece, and adding up the results. In the limit, we get the surface integral

$$\iint_S f\,dS.$$

If S is parametrized by $\vec{r}(u,v)$ for (u,v) in the domain D, this integral becomes

$$\iint_D f(\vec{r}(u,v))\,\underbrace{|\vec{r}_u \times \vec{r}_v|\;du\,dv}_{dS}.$$

Example: If S is the portion of the paraboloid parametrized by $\vec{r}(u,v) = \langle u\cos v, u\sin v, u^2\rangle$ for $0 \le u \le 1$, $0 \le v \le 2\pi$, find $\displaystyle\iint_S \sqrt{4z+1}\,dS$.

$$|\vec{r}_u \times \vec{r}_v| = |\,\langle \cos v, -\sin v, 2u\rangle \times \langle -u\sin v, u\cos v, 0\rangle\,| =$$

$$|\,\big\langle -2u^2\cos v, -2u^2\sin v, u\big\rangle\,| = u\sqrt{4u^2+1};$$

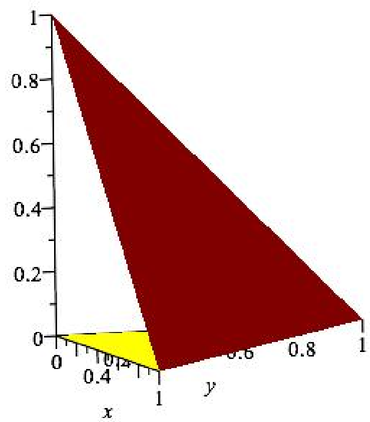
$$dS = \left(u\sqrt{4u^2+1}\right)\,du\,dv;$$

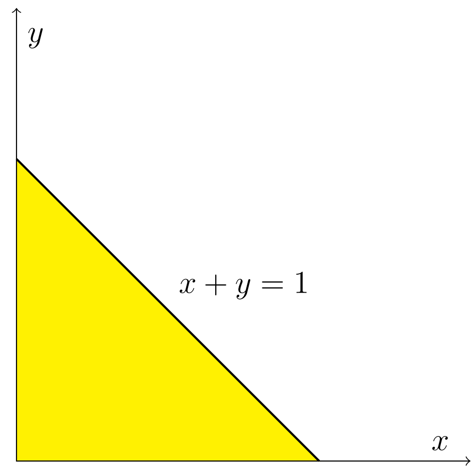
$$\iint_S \sqrt{4z+1}\,dS = \int_0^{2\pi}\int_0^1 \sqrt{4u^2+1}\left(u\sqrt{4u^2+1}\right)\,du\,dv = \int_0^{2\pi}\int_0^1 \left(4u^3+u\right)\,du\,dv = 3\pi.$$

Here are some applications of surface integrals of scalar functions:

1. $\displaystyle\iint_S 1\,dS$ is the surface area of S.
2. If f represents the mass density of the surface at a point (say in grams per square meter), then $\displaystyle\iint_S f\,dS$ is the total mass of the surface.
3. The average value of f on S is $\displaystyle\frac{1}{\text{area}(S)}\iint_S f\,dS$.

Example: If S is the portion of the plane $x+y+z=1$ in the first octant, oriented with unit normal vector $\vec{n}$ slanting upward, and $F(x,y,z) = \langle x, y, z\rangle$, integrate the component of F in the direction of $\vec{n}$ over the surface S.





S has equation $z = 1-x-y$, so we can parametrize S by $\vec{r}(u,v) = \langle u, v, 1-u-v\rangle$. The limits on u and v are the limits on x and y over S, which are $0 \le u \le 1$, $0 \le v \le 1-u$. A normal vector is

$$\vec{r}_u \times \vec{r}_v = \langle 1, 0, -1\rangle \times \langle 0, 1, -1\rangle = \langle 1, 1, 1\rangle\,,$$

which we can check has the correct orientation. Therefore, the unit normal vector is

$$\vec{n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle,$$

and the component of F in this direction is

$$\frac{F\cdot\vec{n}}{|\vec{n}|} = F\cdot\vec{n} = \langle x, y, z\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \frac{x+y+z}{\sqrt{3}}.$$

This gives us

$$\iint_S F\cdot\vec{n}\,dS = \iint_S \frac{x+y+z}{\sqrt{3}}\,dS = \int_0^1\int_0^{1-u} \frac{u+v+(1-u-v)}{\sqrt{3}}\;|\vec{r}_u\times\vec{r}_v|\;dv\,du =$$

$$\int_0^1\int_0^{1-u} \frac{1}{\sqrt{3}}\sqrt{3}\,dv\,du = \int_0^1\int_0^{1-u} 1\,dv\,du = \int_0^1 (1-u)\,du = \frac{1}{2}.$$

In the last example, it was not an accident that the $\sqrt{3}$ in the denominator of $F \cdot \vec{n}$ and the $\sqrt{3}$ in dS canceled out. In general, for any vector field F and any surface S, we have

$$dS = |\vec{r}_u \times \vec{r}_v| \; du\, dv$$

$$\vec{n} = \frac{1}{|\vec{r}_u \times \vec{r}_v|}(\vec{r}_u \times \vec{r}_v)$$

$$F \cdot \vec{n} = F \cdot \left(\frac{1}{|\vec{r}_u \times \vec{r}_v|}(\vec{r}_u \times \vec{r}_v) \right) = (F \cdot (\vec{r}_u \times \vec{r}_v)) \left(\frac{1}{|\vec{r}_u \times \vec{r}_v|} \right)$$

$$F \cdot n\, dS = \left((F \cdot (\vec{r}_u \times \vec{r}_v)) \left(\frac{1}{|\vec{r}_u \times \vec{r}_v|} \right) \right) (|\vec{r}_u \times \vec{r}_v| \; du\, dv) = (F \cdot (\vec{r}_u \times \vec{r}_v)) \; du\, dv.$$

Definition: The surface integral of a vector function F over an oriented surface S is defined to be

$$\iint_S F \cdot \vec{n}\, dS, \text{ also denoted } \iint_S F \cdot d\vec{S},$$

which is evaluated using

$$d\vec{S} = \vec{n}\, dS = (\vec{r}_u \times \vec{r}_v)\, du\, dv.$$

By the same reasoning we applied to $\int_\gamma F \cdot \vec{n}\, ds$ when we were thinking about the vector forms of Green's Theorem, we can see that if F is a fluid flow field, then the surface integral $\iint_S F \cdot \vec{n}\, dS$ represents the rate of flow through the surface S in the direction given by $\vec{n}$.

If F is an electric field, then the surface integral $\iint_S F \cdot \vec{n}\, dS$ represents the electric flux through S. If S is the outward-oriented boundary of a three-dimensional region D, then Gauss's Law says that the electric flux through S is a constant multiple of the net charge on D. (The constant depends on the units, not on D or F.)

If f represents temperature, then an appropriate constant multiple of $-\nabla f$ represents the heat flow field F, and $\iint_S F \cdot \vec{n}\, dS$ represents the rate of heat flow through S.

Example: Let D be the three-dimensional region $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$, let S be the boundary (surface) of D oriented so $\vec{n}$ points outward, and let $F(x, y, z) = \langle -y, x, z \rangle$. Find $\iint_S F \cdot d\vec{S}$.

The surface of S consists of three parts, the cylinder $x^2 + y^2 = 1$ with $0 \leq z \leq 1$, oriented so $\vec{n}$ points away from the z-axis; the disc $z = 1$ with $x^2 + y^2 \leq 1$, oriented so $\vec{n}$ points upward; and the the disc $z = 0$ with $x^2 + y^2 \leq 1$, oriented so $\vec{n}$ points downward.

For the cylindrical surface S_1, we can use the parametrization $\vec{r}(u, v) = \langle \cos u, \sin u, v \rangle$ with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$. Then we have

$$\vec{r}_u \times \vec{r}_v = \langle -\sin u, \cos u, 0 \rangle \times \langle 0, 0, 1 \rangle = \langle \cos u, \sin u, 0 \rangle ,$$

which we can check has the correct orientation. Then

$$\iint_{S_1} \langle -y, x, z \rangle \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 \langle -\sin u, \cos u, v \rangle \cdot \langle \cos u, \sin u, 0 \rangle \; du\, dv = \int_0^{2\pi} \int_0^1 0 \, du\, dv = 0.$$

We could also have figured this out without parametrizing the surface. On the cylinder, the normal vector at point (x, y, z) points directly away from the z-axis, so it points in the direction given by $\langle x, y, 0 \rangle$. This direction is normal to the direction $\langle -y, x, z \rangle$, or of F. We can see this must be the case because we can write F as the sum of two vector fields $\langle -y, x, 0 \rangle + \langle 0, 0, z \rangle$, each of which we can see flows tangent to the cylinder, and we can check that it is the case because $\langle x, y, 0 \rangle \cdot \langle -y, x, z \rangle = 0$. Since F is normal to $\vec{n}$, or parallel to S_1, the component of F in the direction of $\vec{n}$, or normal to S_1, is zero. That is $F \cdot \vec{n} = 0$, so $\iint_{S_1} F \cdot \vec{n} \, dS = 0$.

For the top disc S_2, we can use the parametrization $\vec{r}(u, v) = \langle u, v, 1 \rangle$ with $u^2 + v^2 \leq 1$. Then we have

$$\vec{r}_u \times \vec{r}_v = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle ,$$

which we can see points upward, which is the correct orientation. It is already a unit vector.

$$\iint_{S_2} F \cdot d\vec{S} = \iint_{u^2+v^2 \leq 1} \langle -v, u, 1 \rangle \cdot \langle 0, 0, 1 \rangle \; dA = \iint_{u^2+v^2 \leq 1} 1 \, dA = \pi.$$

We could also have figured this out without parametrizing the disc. Since the disc is horizontal, the unit normal vector is $\hat{k}$, and the component of F in this direction is its z-component, which is z, or 1 on this disc. Integrating 1 over the disc gives its surface area.

For the bottom disc S_3, we can reason that $\vec{n} = -\hat{k}$, and the component of F in this direction is minus its z-component, which is $-z$, or 0 on this disc. The surface integral is 0.

$$\iint_S F \cdot d\vec{S} = \iint_{S_1} F \cdot d\vec{S} + \iint_{S_2} F \cdot d\vec{S} + \iint_{S_3} F \cdot d\vec{S} = 0 + \pi + 0 = \pi.$$

Example: Stokes' Theorem (which we haven't gotten to yet) says: If S is a sufficiently nice oriented surface in $\mathbb{R}^3$ with positively oriented boundary ∂S, and F is a sufficiently nice vector field, then

$$\iint_S (\nabla \times F) \cdot \vec{n}\, dS = \int_{\partial S} F \cdot T\, ds.$$

Verify this when S is the top half of the unit sphere, oriented with $\vec{n}$ pointing up, ∂S is the unit circle in the xy plane, counterclockwise as seen from above, $F(x, y, z) = \langle -yz, xz, 0\rangle$.

To verify this means to evaluate the integrals on both sides of the equation

$$\iint_S (\nabla \times F) \cdot \vec{n}\, dS = \int_{\partial S} F \cdot T\, ds,$$

and see that we get the same answer.

$$\nabla \times F = \langle -x, -y, 2z\rangle .$$

We can use spherical coordinates to parametrize S (with $u = \phi$, $v = \theta$, $\rho = 1$):

$$\langle x, y, z\rangle = \vec{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u\rangle \qquad 0 \le u \le \frac{\pi}{2} \qquad 0 \le v \le 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \langle \cos u \cos v, \cos u \sin v, -\sin u\rangle \times \langle -\sin u \sin v, \sin u \cos v, 0\rangle =$$

$$\left\langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u\right\rangle$$

$$\iint_S (\nabla \times F) \cdot \vec{n}\, dS =$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \langle -\sin u \cos v, -\sin u \sin v, 2\cos u\rangle \cdot \left\langle \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u\right\rangle\, du\, dv =$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} -\sin^3 u \cos^2 v - \sin^3 u \sin^2 v + 2\cos^2 u \sin u\, du\, dv =$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} (-\sin^2 u + 2\cos^2 u) \sin u\, du\, dv = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (-(1 - \cos^2 u) + 2\cos^2 u) \sin u\, du\, dv =$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} (3\cos^2 u - 1) \sin u\, du\, dv = \int_0^{2\pi} (-\cos^3 u + \cos u)\Big|_{u=0}^{u=\frac{\pi}{2}} dv = 0.$$

For the line integral, parametrize ∂S via $\vec{r}(t) = \langle \cos t, \sin t, 0\rangle$ for $0 \le t \le 2\pi$. Then

$$\int_{\partial S} F \cdot T\, ds = \int_{\partial S} F \cdot\, d\vec{r} = \int_0^{2\pi} \langle 0, 0, 0\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt = \int_0^{2\pi} 0\, dt = 0.$$

Using Stokes's Theorem would have been an easier way to evaluate the surface integral!

Example: Find the average z-component of a point on the conical surface given by

$$z = \sqrt{x^2 + y^2} \qquad 0 \leq z \leq 1.$$

Example: If S is the surface

$$z = 1 - x^2 \qquad z \geq 0 \qquad 0 \leq y \leq 1,$$

oriented with $\vec{n}$ pointing upward, and $F(x, y, z) = \langle x, y, z \rangle$, find

$$\iint_S F \cdot d\vec{S}.$$

Example: Suppose that D is a three-dimensional region in $\mathbb{R}^3$ defined by $(x, y) \in E$, $0 \le z \le g(x, y)$, where E is a simply connected region in the xy plane with piecewise smooth boundary, and suppose that $F(x, y, z) = \langle 0, 0, f(x, y)\rangle$ is a continuous vector field. Let S be the boundary (surface) of D, oriented so that $\vec{n}$ is pointing outward (away from D). Then S breaks up into three pieces, S_1 in the xy plane with $\vec{n}$ pointing downward, S_2 in the graph of g with $\vec{n}$ pointing upward, and S_3, a vertical curved surface connecting the boundaries of S_1 and S_2, with $\vec{n}$ pointing horizontally outward away from D.

By parametrizing S_1 with $\vec{r}(u, v) = \langle v, u, 0\rangle$ and S_2 with $\vec{r}(u, v) = \langle u, v, g(u, v)\rangle$ (check that these parametrizations give the correct orientations), and expressing the surface integrals as integrals in u and v, show that

$$\iint_{S_2} F \cdot d\vec{S} = -\iint_{S_1} F \cdot d\vec{S}.$$

Explain why we know that

$$\iint_{S_3} F \cdot d\vec{S} = 0.$$

What is $\iint_S F \cdot d\vec{S}$? Given what we know about F, and the interpretation of $\iint_S F \cdot d\vec{S}$ as the rate of flow of F through S, explain why we should have expected this.

(This is not just because F is vertical. If D is the cylindrical surface $x^2 + y^2 \le 1$, $0 \le z \le 1$, and $F(x, y, z) = \langle 0, 0, z\rangle$, you can check that the surface integral of F over the boundary of D is π.)