Math 11
Fall 2016
Section 1
Wednesday, November 9, 2016

First, some important points from the last class:

If S is parametrized by $\vec{r}(u, v)$ for (u, v) in the domain D, f is a scalar function, and F is a vector field:

$$\iint_S f\,dS = \iint_D f(\vec{r}(u,v))\,\underbrace{|\vec{r}_u \times \vec{r}_v|\;du\,dv}_{dS};$$

$$\iint_S F \cdot \vec{n}\,dS = \iint_S F \cdot d\vec{S} = \iint_D F(\vec{r}(u,v))(\vec{r}_u \times \vec{r}_v)\,du\,dv.$$

$\displaystyle\iint_S 1\,dS$ is the surface area of S.

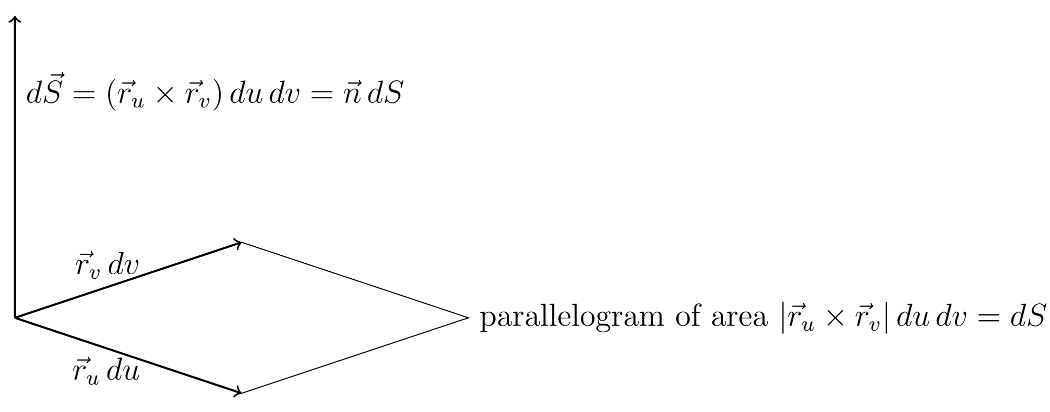
If f represents the mass density of the surface at a point (say in grams per square meter), then $\displaystyle\iint_S f\,dS$ is the total mass of the surface.

The average value of f on S is $\displaystyle\frac{1}{\text{area}(S)}\iint_S f\,dS$.

If F is a fluid flow field, then the surface integral $\displaystyle\iint_S F \cdot \vec{n}\,dS$ represents the rate of flow through the surface S in the direction given by $\vec{n}$.

If F is an electric field, then the surface integral $\displaystyle\iint_S F \cdot \vec{n}\,dS$ represents the electric flux through S.

You can imagine an infinitely small piece of the surface S, which is the image under $\vec{r}$ of an infinitely small rectangle of dimensions $du \times dv$ in the uv plane:



Theorem (Stokes' Theorem): If S is an oriented piecewise-smooth surface, and ∂S, the positively-oriented boundary of S (counterclockwise around S as viewed from the top of a unit normal vector $\vec{n}$), is a simple, closed, piecewise smooth curve, and F is a vector field whose components have continuous partial derivatives on some open region containing S, then

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_{\partial S} F \cdot d\vec{r}.$$

Example: If S is the surface $x^2 + y^2 + 3z^2 = 1$, $z \geq 0$, oriented with $\vec{n}$ pointing upward, and $F(x, y, z) = \langle y, -x, zx^3y^2 \rangle$, evaluate

$$\iint_S (\nabla \times F) \cdot d\vec{S}.$$

The boundary of this surface is the unit circle γ in the xy plane, oriented counterclockwise as viewed from above. We can parametrize γ by $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \leq t \leq 2\pi$, so $d\vec{r} = \langle -\sin t, \cos t, 0 \rangle$. Then, by Stokes' Theorem,

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_\gamma F \cdot d\vec{r} = \int_0^{2\pi} \langle \sin t, -\cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \; dt = \int_0^{2\pi} -1 \, dt = -2\pi.$$

Example: If $F(x,y,z) = \langle x^2, 2xy + x, z\rangle$ and γ is the unit circle in the xy plane, oriented counterclockwise as seen from above, find $\displaystyle\int_\gamma F \cdot d\vec{r}$.

If S is the unit disc in the xy-plane, oriented with unit normal vector pointing upward, then $\gamma = \partial S$, and by Stokes' Theorem we have

$$\int_\gamma F \cdot d\vec{r} = \iint_S (\nabla \times F) \cdot d\vec{S} = \iint_S \langle 0, 0, 2y + 1\rangle \cdot d\vec{S}.$$

We can parametrize S by

$$\vec{r}(u,v) = \langle u, v, 0\rangle \qquad u^2 + v^2 \leq 1 \qquad \vec{r}_u \times \vec{r}_v = \langle 1,0,0\rangle \times \langle 0,1,0\rangle = \langle 0,0,1\rangle .$$

This is the correct orientation. (If it were not, we could interchange u and v in the parametrization, or just multiply our answer by -1.) Then

$$d\vec{S} = \vec{r}_u \times \vec{r}_v \, du \, dv = \langle 0,0,1\rangle \; du \, dv$$

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \iint_{u^2+v^2\leq 1} \langle 0,0,2v+1\rangle \cdot \langle 0,0,1\rangle \; du \, dv = \iint_{u^2+v^2\leq 1} 2v + 1 \, du \, dv.$$

By symmetry we have

$$\iint_{u^2+v^2\leq 1} 2v \, du \, dv = 0.$$

$$\iint_{u^2+v^2\leq 1} 1 \, du \, dv = (\text{area of region } u^2 + v^2 \leq 1) = \pi.$$

Therefore

$$\int_\gamma F \cdot d\vec{r} = \iint_S (\nabla \times F) \cdot d\vec{S} = \iint_{u^2+v^2\leq 1} 2v + 1 \, du \, dv = \pi.$$

We could also do this without parametrizing S: Since S is in the xy plane, oriented with upward pointing normal, we have $\vec{n} = \langle 0,0,1\rangle$, and

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \iint_S \langle 0,0,2y+1\rangle \cdot \vec{n} \, dS = \iint_S \langle 0,0,2y+1\rangle \cdot \langle 0,0,1\rangle \; dS =$$

$$\iint_S 2y + 1 \, dS = \underbrace{\iint_S 2y \, dS}_{\text{0 by symmetry}} + \iint_S 1 \, dS = 0 + \text{area}(S) = \pi.$$

Example: Suppose that F is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives everywhere, and when $z = 0$ we have

$$(\nabla \times F)(x, y, 0) = \langle 0, 0, 3 \rangle \,.$$

Let S be the top half of the unit sphere, oriented with $\vec{n}$ pointing up. Find

$$\iint_S (\nabla \times F) \cdot d\vec{S}.$$

The boundary of S is γ, the unit circle in the xy-plane oriented counterclockwise as viewed from above. By Stokes' Theorem, we have

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_\gamma F \cdot d\vec{r}.$$

Let D be the unit disc in the xy-plane, oriented with $\vec{n}$ pointing up. Then γ is also the boundary of D, and by Stokes' Theorem, we have

$$\int_\gamma F \cdot d\vec{r} = \iint_D (\nabla \times F) \cdot d\vec{S}.$$

On D, we have $\nabla \times F = \langle 0, 0, 3 \rangle$ and $\vec{n} = \langle 0, 0, 1 \rangle$, so

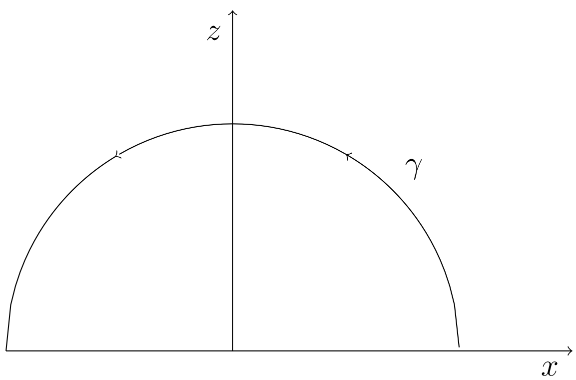
$$\iint_D (\nabla \times F) \cdot d\vec{S} = \iint_D (\nabla \times F) \cdot \vec{n} \, dS = \iint_D \langle 0, 0, 3 \rangle \cdot \langle 0, 0, 1 \rangle \, dS = \iint_D 3 \, dS = 3\pi.$$

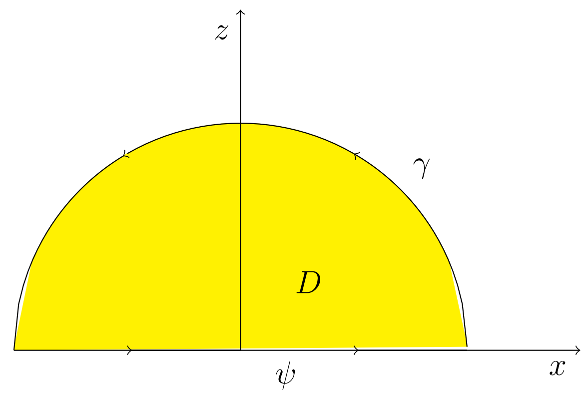
Putting this all together,

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_\gamma F \cdot d\vec{r} = \iint_D (\nabla \times F) \cdot d\vec{S} = 3\pi.$$

Example: Evaluate $\int_\gamma F \cdot d\vec{r}$ where $F(x,y,z) = \langle yz, xz, xy \rangle$ and γ is the portion of the circle $x^2 + z^2 = 1$, $y = 1$ above the xy plane, oriented from $(1,1,0)$ to $(-1,1,0)$.

In the plane $y = 1$:





Using Stokes' Theorem: Let ψ be the line segment from $(-1,1,0)$ to $(1,1,0)$.

Then $\gamma + \psi$ is the boundary of D, the half-disc $x^2 + z^2 \leq 1$, $z \geq 0$, $y = 1$, oriented appropriately. (In the picture, $\vec{n}$ should be pointing at us, which gives $\vec{n} = \langle 0, -1, 0 \rangle$.) By Stokes's Theorem

$$\int_\gamma F \cdot d\vec{r} + \int_\psi F \cdot d\vec{r} = \iint_D (\nabla \times F) \cdot d\vec{S}.$$

We can compute $\nabla \times F = \langle 0,0,0 \rangle$, so

$$\iint_D (\nabla \times F) \cdot d\vec{S} = \iint_D \langle 0,0,0 \rangle \cdot d\vec{S} = 0.$$

On ψ, we have $z = 0$, so $F = \langle 0, 0, xy \rangle$. Also, on ψ we have $d\vec{r} = \vec{T}\, ds$ and $\vec{T} = \langle 1, 0, 0 \rangle$, so

$$\int_\psi F \cdot d\vec{r} = \int_\psi F \cdot \vec{T}\, ds = \int_\psi \langle 0, 0, xy \rangle \cdot \langle 1, 0, 0 \rangle \; ds = 0.$$

Therefore, we have

$$\int_\gamma F \cdot d\vec{r} + 0 = 0;$$

$$\int_\gamma F \cdot d\vec{r} = 0.$$

Example continued with some other methods:

Evaluate $\int_\gamma F \cdot d\vec{r}$ where $F(x,y,z) = \langle yz, xz, xy\rangle$ and γ is the portion of the circle $x^2+z^2=1$, $y=1$ above the xy plane, oriented from $(1,1,0)$ to $(-1,1,0)$.

Using the Fundamental Theorem of Line Integrals: The fact that $\nabla \times F = \langle 0,0,0\rangle$ on all of $\mathbb{R}^3$ tells us that F is conservative. If we look for a potential function for F we will discover that $F = \nabla f$ where $f(x,y,z) = xyz$. Therefore

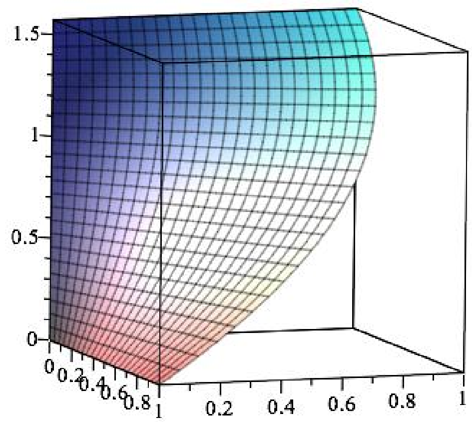
$$\int_\gamma F \cdot d\vec{r} = \int_\gamma \nabla f \cdot d\vec{r} = f(-1,1,0) - f(1,1,0) = 0.$$

Directly: Parametrize γ by $\vec{r}(t) = \langle \cos t, 1, \sin t\rangle$ for $0 \le t \le \pi$. Then

$$\int_\gamma \langle yz, xz, xy\rangle \cdot d\vec{r} = \int_0^\pi \langle \sin t, \cos t \sin t, \cos t\rangle \cdot \langle -\sin t, 0 \cos t\rangle\ dt = \int_0^\pi (\cos^2 t - \sin^2 t)\, dt =$$

$$\int_0^\pi \left(\frac{1+\cos(2t)}{2} - \frac{1-\cos(2t)}{2}\right)\ dt = \int_0^\pi \cos(2t)\, dt = \left.\frac{\sin(2t)}{2}\right|_{t=0}^{t=\pi} = 0.$$

Example: S is the surface parametrized by $\vec{r}(u, v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \dfrac{\pi}{2}$, oriented so $\vec{n}$ points in the direction of $\vec{r}_u \times \vec{r}_v$, and γ is the positively oriented boundary of S.



Determine which way the unit vector $\vec{n}$ points, and draw arrows on the picture to indicate the orientation of γ.

If $F(x, y, z) = \langle z, x^2, y\rangle$, find

$$\int_\gamma F \cdot d\vec{r}$$

in two ways, by directly computing the line integral, and by using Stokes' Theorem.

Example: Use Stokes's Theorem to find $\iint_S (\nabla \times F)\, d\vec{S}$ where S is the portion of the cone $z = 1 - \sqrt{x^2 + y^2}$ in the first octant, oriented so $\vec{n}$ points away from the z-axis, and $F(x, y, z) = \left\langle -y + (x + z)\cos x, x - zy^3, e^{xy} \right\rangle$.

Example: Suppose that F is a vector field whose components have continuous partial derivatives, and S is a sphere with outward-pointing normal. Use Stokes' Theorem to show that

$$\iint_S (\nabla \times F) \cdot d\vec{S} = 0.$$

Challenge: Suppose that F is a fluid flow field that varies over time, so that $F(x, y, z, t)$ is the velocity of fluid flow at point (x, y, z) and time t. If S is an oriented surface, how would you use integrals to find how much fluid flows through S between times $t = a$ and $t = b$?