Math 11
Fall 2016
Section 1
Friday, November 11, 2016

First, some important points from the last class:

Theorem (Stokes' Theorem): If S is an oriented piecewise-smooth surface, and ∂S, the positively-oriented boundary of S (counterclockwise around S as viewed from the top of a unit normal vector $\vec{n}$), is a simple, closed, piecewise smooth curve, and F is a vector field whose components have continuous partial derivatives on some open region containing S, then

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_{\partial S} F \cdot d\vec{r}.$$

Stokes's Theorem is one multivariable version of the Fundamental Theorem of Calculus. It says that the integral of the curl (rotational tendency) of F over a surface S equals the line integral (circulation) of F around the boundary of S.

We can use Stokes' Theorem to simplify the evaluation of integrals in several ways:

by evaluating $\iint_S (\nabla \times F) \cdot d\vec{S}$ instead of $\int_{\partial S} F \cdot d\vec{r}$, when this is easier;

by evaluating $\int_{\partial S} F \cdot d\vec{r}$ instead of $\iint_S (\nabla \times F) \cdot d\vec{S}$, when this is easier;

by evaluating $\iint_{S_1} (\nabla \times F) \cdot d\vec{S}$ instead of $\iint_S (\nabla \times F) \cdot d\vec{S}$, when this is easier and the two surfaces have the same boundary;

and in other ways.

Definition: In $\mathbb{R}^3$, the solid region E is Type I if it is given by

$$u_1(x, y) \leq z \leq u_2(x, y) \qquad (x, y) \in D$$

where u_1 and u_2 are continuous functions and D is a Type I or Type II region in the plane. Type II and Type III are defined similarly.

E is simple if it is Type I, Type II, and Type III.

Example: The region inside a sphere is simple. The region between two spheres centered at the origin is not simple, but it can be divided into eight simple regions, one in each octant.

Theorem (Divergence Theorem): If E is a simple solid region or can be divided into finitely many simple solid regions, ∂E is the positively-oriented boundary of E (with $\vec{n}$ pointing outward from E), and F is a vector field whose components have continuous partial derivatives on some open region containing S, then

$$\iint_S F \cdot d\vec{S} = \iiint_E \nabla \cdot F \, dV.$$

The Divergence Theorem (or Gauss's Theorem) is one multivariable version of the Fundamental Theorem of Calculus. It says that the integral of the divergence (expansionary tendency) of F over a solid region E equals the surface integral (rate of flow) of F across the boundary of E.

Example: Find the flux of the vector field $F(x, y, z) = \langle -y, x, z \rangle$ outward through the unit sphere.

The unit sphere S is the boundary of the solid unit ball E, so by the Divergence Theorem,

$$\iint_S F \cdot d\vec{S} = \iiint_E \nabla \cdot F \, dV = \iiint_E 1 \, dV = \text{volume}(E) = \frac{4\pi}{3}.$$

Example: (the same example continued)

We could have computed the flux of the vector field $F(x,y,z) = \langle -y, x, z\rangle$ outward through the unit sphere directly, by parametrizing the unit sphere. If we use cylindrical coordinates, in which the unit sphere has the equation $r^2 + z^2 = 1$, we can set $v = \theta$ and $u = z$ to get

$$\langle x,y,z\rangle = \vec{r}(u,v) = \left\langle \sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u \right\rangle \qquad -1 \le u \le 1 \qquad 0 \le v \le 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \left\langle \frac{-u}{\sqrt{1-u^2}}\cos v, \frac{-u}{\sqrt{1-u^2}}\sin v, 1 \right\rangle \times \left\langle -\sqrt{1-u^2}\sin v, \sqrt{1-u^2}\cos v, 0 \right\rangle =$$

$$\left\langle -\sqrt{1-u^2}\cos v, -\sqrt{1-u^2}\sin v, -u \right\rangle = \langle -x, -y, -z\rangle\,,$$

which has the wrong orientation. To account for this, we change the sign:

$$\iint_S \langle -y, x, z\rangle \cdot d\vec{S} =$$

$$\int_0^{2\pi}\int_{-1}^{1} \left\langle -\sqrt{1-u^2}\sin v, \sqrt{1-u^2}\cos v, u \right\rangle \cdot \left\langle \sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u \right\rangle du\,dv =$$

$$\int_0^{2\pi}\int_{-1}^{1} u^2\,du\,dv = \int_0^{2\pi} \frac{2}{3}\,dv = \frac{4\pi}{3}.$$

We could also have noticed that on the unit sphere, the vector $\langle x,y,z\rangle$ is a unit vector pointing away from the origin, and therefore normal to the sphere. That is, $\langle x,y,z\rangle = \vec{n}$. Therefore we could write

$$\iint_S F\cdot d\vec{S} = \iint_S F\cdot \vec{n}\,dS = \iint_S \langle -y,x,z\rangle \cdot \langle x,y,z\rangle\; dS = \iint_S z^2\,dS.$$

Evaluating this integral by parametrizing the sphere in cylindrical coordinates gives pretty much the same integral we just evaluated.

Example: Find the flux of the vector field $F(x,y,z) = \langle xe^y + \sin z, -e^y - 9z^2y, 3z^3 + yz\rangle$ across the surface S, the top half of the unit sphere ($z \geq 0$), oriented so $\vec{n}$ points upward.

Since the divergence of F is much simpler than F itself,

$$\nabla \cdot F = (e^y) + (-e^y - 9z^2) + (9z^2 + y) = y,$$

we would like to find a way to use the Divergence Theorem. We can do this by combining S with another surface to form the boundary of a solid region. Let S_1 be the unit disc in the xy-plane, oriented with $\vec{n}$ pointing downward. Then $S + S_1$ is the boundary of a solid region E, which is the portion of the unit ball above the xy plane. By the Divergence Theorem,

$$\iiint_E \nabla \cdot F \, dV = \iint_S F \cdot d\vec{S} + \iint_{S_1} F \cdot d\vec{S}.$$

Evaluating the two integrals that are not the one we are trying to avoid evaluating directly:

$$\iiint_E \nabla \cdot F \, dV = \iiint_E y \, dV = 0 \text{ (by symmetry);}$$

$$\iint_{S_1} F \cdot d\vec{S} = \iint_{S_1} F \cdot \vec{n} \, dS = \iint_{S_1} \left\langle xe^y + \sin(0), -e^y - 9(0)^2y, 3(0)^3 + (y)(0)\right\rangle \cdot \langle 0,0,-1\rangle \; dS =$$

$$\iint_{S_1} \langle xe^y, -e^y, 0\rangle \cdot \langle 0,0,-1\rangle \; dS = \iint_{S_1} 0 \, dS = 0.$$

Going back to our equation:

$$\underbrace{\iiint_E \nabla \cdot F \, dV}_{0} = \iint_S F \cdot d\vec{S} + \underbrace{\iint_{S_1} F \cdot d\vec{S}}_{0};$$

$$\iint_S F \cdot d\vec{S} = 0.$$

Example: S is the portion of the plane $x+y+z=1$ in the first octant, oriented so $\vec{n}$ points upward. Find the flux of the vector field $\langle x,y,z\rangle$ through S.

Via the Divergence Theorem: Let E be the corner of the first octant cut off by that plane. Then the boundary of E consists of S, and surfaces S_1, S_2 and S_3 contained in the coordinate planes. The Divergence Theorem tells us

$$\iiint_E \nabla\cdot F\,dV = \iint_{\partial E} F\cdot d\vec{S} = \iint_S F\cdot d\vec{S} + \iint_{S_1} F\cdot d\vec{S} + \iint_{S_2} F\cdot d\vec{S} + \iint_{S_3} F\cdot d\vec{S}.$$

If S_1 lies in the xy plane $z=0$, on S_1 we have $F=\langle x,y,z\rangle=\langle x,y,0\rangle$ and $\vec{n}=\langle 0,0,-1\rangle$, so

$$\iint_{S_1} F\cdot\vec{S} = \iint_{S_1} F\cdot\vec{n}\,dS = \iint_{S_1} \langle x,y,0\rangle\cdot\vec{n}\,\langle 0,0,-1\rangle\,dS = 0.$$

By the same reasoning,

$$\iint_{S_2} F\cdot\vec{S} = \iint_{S_3} F\cdot\vec{S} = 0.$$

Now we have

$$\iint_S F\cdot d\vec{S} = \iiint_E \nabla\cdot F\,dV = \iiint_E 3\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 3\,dz\,dy\,dx =$$

$$3\int_0^1\int_0^{1-x} 1-x-y\,dy\,dx = 3\int_0^1 \left((1-x)y-\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1-x} dx = \frac{3}{2}\int_0^1 (1-x)^2\,dx =$$

$$\frac{3}{2}\left(-\frac{(1-x)^3}{3}\right)\Bigg|_{x=0}^{x=1} = \frac{1}{2}.$$

Directly: Parametrize S via

$$\vec{r}(u,v)=\langle u,v,1-u-v\rangle \qquad 0\le u\le 1 \qquad 0\le v\le 1-u;$$

$$\vec{r}_u\times\vec{r}_v = \langle 1,0,-1\rangle\times\langle 0,1,-1\rangle = \langle 1,1,1\rangle.$$

This has the correct orientation.

$$\iint_S F\cdot d\vec{S} = \int_0^1\int_0^{1-u} \langle u,v,1-u-v\rangle\cdot\langle 1,1,1\rangle\,dv\,du = \int_0^1\int_0^{1-u} 1\,dv\,du =$$

$$\int_0^1 (1-u)\,du = \frac{1}{2}.$$

Example: The electric or gravitational field produced by a point charge or mass at point P is directed directly toward or away from P, and its magnitude is inversely proportional to the square of the distance from P. If we denote the distance from P by d, and let k be a suitable constant of proportionality (possibly negative), we can say $F = \dfrac{k}{d^2}\,\vec{u}$, where $\vec{u}$ is a unit vector pointing directly away from P. (k is a constant multiple of the amount of charge or mass at P, depending on physical constants and the units of measurement.)

If S is a sphere of radius a centered at P, with outward pointing normal, then we can evaluate the flux of F outward across S: The unit normal vector $\vec{n}$ points directly away from P, so we can write $F = \dfrac{k}{d^2}\,\vec{n}$. On this sphere, $d = a$, so $F = \dfrac{k}{a^2}\,\vec{n}$. This gives

$$\iint_S F \cdot \vec{n}\,dS = \iint_S \left(\frac{k}{a^2}\,\vec{n}\right) \cdot \vec{n}\,dS = \iint_S \frac{k}{a^2}\,dS = \frac{k}{a^2}\text{area}(S) = \frac{k}{a^2}(4\pi a^2) = 4\pi k.$$

Notice that this answer is the same no matter what value we choose for a.

We can check that $\nabla \cdot F = 0$, except at P where F is undefined. This lets us compute the flux of F outward across the boundary S of any solid region E, as long as P does not lie on S: If P is not in E, then the divergence of F throughout E is zero, so the flux of F across S is zero.

If P is in E, we can let S_1 be a very small sphere centered at P, with inward pointing normal, and let E_1 be the part of E that is not enclosed by S_1. Because S_1 has inward-pointing normal,

$$\iint_{S_1} F \cdot \vec{S} = -4\pi k.$$

The boundary of E_1 is $S + S_1$, so by the Divergence Theorem,

$$\iint_S F \cdot d\vec{S} + \iint_{S_1} F \cdot d\vec{S} = \iiint_{E_1} \nabla \cdot F\,dV = 0;$$

$$\iint_S F \cdot d\vec{S} = -\iint_{S_1} F \cdot d\vec{S} = 4\pi k.$$

That is, if the field F is produced by a point mass or charge, then the flux of F across S is a constant $C = 4\pi k$ times the amount of mass or charge inside S.

The same thing holds even if the field is produced by multiple point masses or charges, since we just add the individual fields together.

The same thing also holds if the field is produced by some continuous distribution of charge or mass. (We can think of dividing an object into many tiny pieces, treating each as a point mass or charge, and taking a limit as the size of the pieces approaches zero.) In the case of charge, this Gauss's Law: If F is an electric field produced by a charge distribution, then the flux of F outward across the boundary S of a three-dimensional region E is a constant C times the total charge on E.

By the Divergence Theorem, the flux is also the integral over E of the divergence $\nabla \cdot F$. If we look at a small enough region so $\nabla \cdot F$ is essentially constant, this is $\nabla \cdot F$ times the volume of the region. Assuming the region is small enough so charge density is essentially constant, we can say

$$C(\text{charge density})(\text{volume}) = C(\text{ total charge}) = C \iint_S F \cdot d\vec{S} \approx C(\nabla \cdot F)\text{volume}$$

That is, the divergence $\nabla \cdot F$ (times the constant C) gives us charge density:

Summary:

If F is an electric field produced by a charge distribution, then $\nabla \cdot F$ is a constant C times the charge density at a point, the total charge on a solid region E is the integral over E of the charge density function $\dfrac{\nabla \cdot F}{C}$, and the flux of F outward through the boundary S of E is

$$\iint_S F \cdot d\vec{S} = \iiint_E \nabla \cdot F \, dV = C \iint_E \frac{\nabla \cdot F}{C} \, dV =$$

$$C \iint_E \text{charge density } \, dV = C(\text{total charge on } E) = C(\text{total charge inside } S).$$

Integrals of scalar functions:

Over an interval on the real line, a two-dimensional region in the plane or a three-dimensional region in space:

$$\int_a^b f(x)\,dx.$$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Along a curve in two- or three-dimensional space:

$$\int_\gamma f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt.$$

$$\int_\gamma f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt.$$

Over a surface in three-dimensional space:

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA.$$

For parametrized curves and surfaces, $|\vec{r}\,'(t)|$ and $|\vec{r}_u \times \vec{r}_v|$ are stretching factors, going from length or area in the domain to length or area in the curve or surface.

Intuitively, the integral of f over some region is a way of multiplying the value of f times the size (length, area, volume) of the region, when the value of f is not constant. The integral is, in fact, the product of the average value of f on the region and the size of the region.

Integrate 1 to get the size (length, area, volume) of the region of integration.

Integrate a mass [charge...] density function to get total mass [charge...].

To find the average value of f on the region of integration, divide the integral of f by the size of the region.

Some of these integrals also have interpretations as area under a curve, or volume under a surface.

Integrals of vector functions:

Integrate the tangential component along a curve in the plane ($F = \langle P, Q \rangle$) or a curve in space ($F = \langle P, Q, R \rangle$):

$$\int_\gamma F \cdot d\vec{r} = \int_\gamma F \cdot \vec{T}\, ds = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_\gamma P\, dx + Q\, dy.$$

$$\int_\gamma F \cdot d\vec{r} = \int_\gamma F \cdot \vec{T}\, ds = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_\gamma P\, dx + Q\, dy + R\, dz.$$

If F is a force, this is the work done by F on an object moving along γ.

Integrate the normal component across a curve in the plane ($F = \langle P, Q \rangle$) or a surface in space ($F = \langle P, Q, R \rangle$):

$$\int_\gamma F \cdot \vec{n}\, ds = \int_\gamma P\, dy - Q\, dx.$$

$$\iint_S F \cdot d\vec{S} = \iint_S F \cdot \vec{n}\, dS = \iint_D F(\vec{r}(u,v)) \times (\vec{r}_u \times \vec{r}_v)\, du\, dv.$$

If F is a fluid flow field, this is the rate of flow of F across the curve or through the surface.

Intuitively, again, these integrals are a way of multiplying the tangential component of F or the normal component of F times the size (length, area, volume) of the region, when this component of F is not constant.

Example: Evaluate $\displaystyle\iint_{\partial E} F \cdot d\vec{S}$, where $F(x, y, z) = \langle yz,\, x^2,\, xz\rangle$ and E is each of the following regions:

1. $x^2 + y^2 \leq z \leq 1$;
2. $x^2 + y^2 \leq z \leq 1$, $x \geq 0$;
3. $x^2 + y^2 \leq z \leq 1$, $x \leq 0$.

Try to do as little actual computation as possible.

Example: Show that if E is a region to which the Divergence Theorem applies, then the volume of E is

$$\frac{1}{3} \iint_{\partial E} \langle x, y, z \rangle \cdot d\vec{S}.$$

Use a computation from earlier in these notes to find the volume of the corner of the first octant cut off by the plane $x + y + z = 1$.

Express the volume of the region above the cone $z = a\sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = b^2$, where a and b are positive constants, using a surface integral. Evaluate this surface integral, using geometric reasoning as much as possible.

Example Suppose that $f(x, y, z)$ is a function satisfying Laplace's equation,

$$\nabla \cdot \nabla f = 0.$$

Show that if E is a solid region to which the Divergence Theorem applies, then

$$\iint_{\partial E} \nabla f \cdot d\vec{S} = 0.$$

If f is a temperature distribution function, then the heat flow is given by $F = -k\nabla f$, where k is some positive constant. This is because, physically, heat flows from areas of high temperature to areas of low temperature, and the rate of heat transfer is proportional to the temperature differential. Use the fact you proved above to explain why solutions to Laplace's equation represent stable (not changing over time) temperature distributions.

Suggestion: Suppose the temperature at point P is falling over time. If S is a tiny sphere centered at P, would you expect the rate of heat flow outward through S to be positive, negative, or zero?