Math 11
Fall 2016
Section 1
Monday, November 14, 2016

First, some important points from the last class:

Definition: In $\mathbb{R}^3$, the solid region E is Type I if it is given by

$$u_1(x, y) \leq z \leq u_2(x, y) \qquad (x, y) \in D$$

where u_1 and u_2 are continuous functions and D is a Type I or Type II region in the plane. Type II and Type III are defined similarly.

E is simple if it is Type I, Type II, and Type III.

Example: The region inside a sphere is simple. The region between two spheres centered at the origin is not simple, but it can be divided into eight simple regions, one in each octant.

Theorem (Divergence Theorem): If E is a simple solid region or can be divided into finitely many simple solid regions, ∂E is the positively-oriented boundary of E (with $\vec{n}$ pointing outward from E), and F is a vector field whose components have continuous partial derivatives on some open region containing S, then

$$\iint_S F \cdot d\vec{S} = \iiint_E \nabla \cdot F \, dV.$$

The Divergence Theorem (or Gauss's Theorem) is one multivariable version of the Fundamental Theorem of Calculus. It says that the integral of the divergence (expansionary tendency) of F over a solid region E equals the surface integral (rate of flow) of F across the boundary of E.

We can use the Divergence Theorem to simplify computations of vectors in at least three ways:

Evaluate $\iint_{\partial E} F \cdot d\vec{S}$ instead of $\iiint_E \nabla \cdot F \, dV$, when this is easier.

Evaluate $\iiint_E \nabla \cdot F \, dV$ instead of $\iint_{\partial E} F \cdot d\vec{S}$, when this is easier.

Evaluate $\iint_S F \cdot d\vec{S}$, when S is not the boundary of a solid region E, by adding another surface S_1 to S to get the boundary of a solid region.

A quick summary of some of the things we have covered:

Vectors and three-dimensional space:

Vectors, and vector operations such as addition, scalar multiplication, dot product, and cross product can be viewed algebraically, geometrically, and in terms of applications.

The geometric interpretation of the dot product allows us to compute scalar projections, vector projections, and work.

The geometric interpretation of the cross product allows us to find normal vectors, areas, and volumes, and to compute torque.

We can use vectors to come up with equations for lines and planes in three dimensions.

Vector functions $\vec{r} : \mathbb{R} \to \mathbb{R}^n$:

Vector functions parametrize curves, and model motion.

The derivative of a vector function lets us compute velocity and the unit tangent vector, and can be used to find a linear approximation.

We use an integral to compute arc length. The same idea allows us to integrate a scalar function along a curve.

Functions of several variables $f : \mathbb{R}^n \to \mathbb{R}$:

Graphs of functions of several variables, and limits of functions of several variables, require new considerations beyond single-variable calculus, such as contour maps, and approaching a point along different paths.

Computing partial derivatives involves no new techniques, but the geometric interpretation is somewhat different. In particular, a function can have partial derivatives without being differentiable.

The gradient ∇f, also called the total derivative when a function is differentiable, has several important interpretations. It tells us something about the shape of a graph, allows us to compute partial derivatives, and gives us a normal vector to a level curve or level surface.

The chain rule can be written using partial derivatives, or using the gradient.

Uses of derivatives of functions of several variables include linear approximations, and techniques to solve max/min problems.

Integrals of functions of several variables require geometric visualization.

Change of variable, in particular polar, cylindrical, and spherical coordinates, can help evaluate integrals.

Functions $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$:

These functions are used to parametrize surfaces.

We use partial derivatives to find a normal vector to a surface, and to define the surface area integral.

The idea behind surface area lets us integrate a scalar function over a surface.

Vector fields $F : \mathbb{R}^n \to \mathbb{R}^n$:

Vector fields model fluid flow and heat flow, force fields, electrical fields, and magnetic fields.

Partial derivatives are used to compute the curl (rotational tendency $\nabla \times F$) and divergence (expansionary tendency $\nabla \cdot F$) of a vector field.

We can integrate vector functions along curves (one interpretation is work) and across curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$ (one interpretation is rate of flow).

Multivariable versions of the Fundamental Theorem of Calculus:

In all cases, suitable smoothness and continuity assumptions apply:

The Fundamental Theorem of Line Integrals:

$$\int_\gamma \nabla f \cdot d\vec{r} = f(\text{end point}) - f(\text{start point}).$$

Green's Theorem:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} \langle P, Q \rangle \cdot d\vec{r}$$

Vector Forms of Green's Theorem ($F = \langle P(x,y), Q(x,y), 0 \rangle$):

$$\int_{\partial D} F \cdot T\, ds = \int_{\partial D} \langle P, Q \rangle \cdot \langle dx, dy \rangle = \int_{\partial D} P\, dx + Q\, dy = \iint_D \text{curl}(F) \cdot \mathbf{k}\, dA.$$

$$\int_{\partial D} F \cdot \vec{n}\, ds = \int_{\partial D} \langle P, Q \rangle \cdot \langle dy, -dx \rangle = \int_{\partial D} (-Q)\, dx + P\, dy = \iint_D \text{div}(F)\, dA.$$

Stokes' Theorem:

$$\iint_S \text{curl}(F) \cdot d\vec{S} = \int_{\partial S} F \cdot d\vec{r}.$$

Divergence Theorem:

$$\iint_S F \cdot d\vec{S} = \iiint_E \text{div}(F)\, dV.$$

Properties of some vector fields:

(1.) F is conservative ($F = \nabla f$) on D.
(2.) Line integrals of F on D are path-independent.
(3.) Line integrals of F around closed curves in D are zero.
(4.) The curl of F is zero on D.

Under suitable smoothness and differentiability assumptions:

$$(1) \iff (2) \iff (3) \implies (4)$$

and if D is simply connected

$$(1) \iff (2) \iff (3) \iff (4).$$