Math 11

Fall 2016

Section 1

Friday, September 16, 2016

First, some important points from the last class:

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|\cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

$$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The scalar projection of $\vec{F}$ onto $\vec{d}$ is

$$proj_{\vec{d}}(\vec{F}) = |\vec{F}|\cos(\theta) = \boxed{\frac{\vec{F} \cdot \vec{d}}{|\vec{d}|}}$$

and the vector projection of $\vec{F}$ onto $\vec{d}$ is

$$\overrightarrow{proj}_{\vec{d}}(\vec{F}) = \left(\frac{|\vec{F}|\cos(\theta)}{|\vec{d}|}\right)\vec{d} = \boxed{\left(\frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right)\vec{d}}$$

The work done by force $\vec{F}$ on an object moving in a straight line with displacement $\vec{d}$ is

$$\boxed{W = \vec{F} \cdot \vec{d}}$$

"Work equals force dot displacement."

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

$$\vec{v} \cdot (\vec{w} + \vec{u}) = (\vec{v} \cdot \vec{w}) + (\vec{v} \cdot \vec{u})$$

$$\vec{v} \cdot (\vec{w} - \vec{u}) = (\vec{v} \cdot \vec{w}) - (\vec{v} \cdot \vec{u})$$

$$(t\vec{v}) \cdot \vec{w} = t(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (t\vec{w})$$

$$\vec{0} \cdot \vec{v} = 0$$

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

Definition: The cross product, or vector product, of vectors $\vec{r}$ and $\vec{F}$ is the vector $\vec{r} \times \vec{F}$ with the following properties:

1. $|\vec{r} \times \vec{F}| = |\vec{r}|\,|\vec{F}|\sin(\theta)$ where θ is the angle between $\vec{r}$ and $\vec{F}$.
2. $\vec{r} \times \vec{F}$ is perpendicular to both $\vec{r}$ and $\vec{F}$.
3. $\vec{r}$, $\vec{F}$ and $\vec{r} \times \vec{F}$ are oriented according to the right-hand rule: If all three vectors are drawn from the same point, and you are looking down from the top of $\vec{r} \times \vec{F}$, rotating from $\vec{r}$ around to $\vec{F}$ appears as a counterclockwise rotation.

$|\vec{r} \times \vec{F}|$ is the area of the parallelogram with edges $\vec{r}$ and $\vec{F}$.

$$\langle v_1, v_2, v_3 \rangle \times \langle w_1, w_2, w_3 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Theorem:

$$\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$$

$$t(\vec{v} \times \vec{w}) = t\vec{v} \times \vec{w} = \vec{v} \times t\vec{w}$$

$$\vec{v} \times (\vec{w} + \vec{u}) = (\vec{v} \times \vec{w}) + (\vec{v} \times \vec{u})$$

$$\vec{v} \times (\vec{w} \times \vec{u}) = (\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}$$

Warning: The cross product is *not* associative.

Definition: The *triple product* of $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$, and $\vec{u} = \langle u_1, u_2, u_3 \rangle$, in that order, is

$$\vec{v} \cdot (\vec{w} \times \vec{u}) = (\vec{v} \times \vec{w}) \cdot \vec{u} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

Theorem: The absolute value of the triple product of $\vec{v}$, $\vec{w}$, and $\vec{u}$ is the volume of the parallelepiped with edges $\vec{v}$, $\vec{w}$, and $\vec{u}$.

The triple product is positive if $\vec{v}$, $\vec{w}$, and $\vec{u}$ are oriented according to the right hand rule in the same way as $\hat{i}$, $\hat{j}$, and $\hat{k}$. It is negative otherwise.

Warm-up Questions:

Suppose an object starts at point (a, b, c) and moves with constant velocity $\vec{v} = \langle x_v,\ y_v,\ z_v\rangle$ for t seconds.

What is its final position?

Suggestion: Find the following:

A unit vector in the direction of the object's motion: $\dfrac{1}{|\vec{v}|}\vec{v}$

The object's speed is $|\vec{v}|$, the magnitude of the velocity vector.

The distance the object travels: (time)(speed) $= t|\vec{v}|$

The object's displacement: (distance)(direction vector) $= (t|\vec{v}|)\left(\dfrac{1}{|\vec{v}|}\vec{v}\right) = t\vec{v}$

(Finally) the object's final position vector:
(initial position vector) + (displacement) $= \langle a, b, c\rangle + t\vec{v} = \langle a + x_v t,\ b + y_v t, c + z_v t\rangle$

What geometric object does the set of all points of the form $(a + x_v t,\ b + y_v t,\ c + z_v t)$ describe? (Be as specific as you can. For example, if the object were a sphere, a complete answer would also identify the center and radius.)

A line through the point (a, b, c) parallel to the vector $\vec{v} = \langle x_v,\ y_v,\ z_v\rangle$. Note we can also write

$$\underbrace{\langle x, y, z\rangle}_{\text{general point on line}} = \underbrace{\langle a, b, c\rangle}_{\text{given point on line}} + t \underbrace{\langle x_v, y_v, z_v\rangle}_{\text{vector parallel to line}}.$$

What does the equation

$$\langle 1, 2, -1\rangle \cdot \langle x, y, z\rangle = 0$$

say about the position vector of the point (x, y, z)?

It is normal (orthogonal, perpendicular) to the vector $\langle 1, 2, -1\rangle$.

What geometric object does the equation

$$x + 2y - z = 0$$

describe?

A plane containing the origin and perpendicular to the vector $\langle 1, 2, -1\rangle$.

Definition: A *vector parametric equation* for the line parallel to vector $\vec{v} = \langle x_v, y_v, z_v \rangle$ passing through the point (x_0, y_0, z_0) with position vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is

$$\vec{r} = \vec{r_0} + t\vec{v}.$$

Here $\vec{r}$ is a variable vector, $\vec{r} = \langle x, y, z \rangle$, so this is

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle x_v, y_v, z_v \rangle .$$

Scalar parametric equations for this line are

$$x = x_0 + tx_v \qquad y = y_0 + ty_v \qquad z = z_0 + tz_v.$$

The variable t is a *parameter*. Different values of the parameter give different points on the line.

As another example,

$$x = \cos\theta \qquad y = \sin\theta$$

are parametric equations for the unit circle in $\mathbb{R}^2$. Different values of the parameter θ give different points on the unit circle.

Example: Find a vector parametric equation for the line through the points $(1, 1, 1)$ and $(2, 3, 4)$.

Point on line: $(1, 1, 1)$
Vector in direction of line: $\langle 2, 3, 4 \rangle - \langle 1, 1, 1 \rangle = \langle 1, 2, 3 \rangle$.
Equation: $\vec{r} = \langle 1, 1, 1 \rangle + t \langle 1, 2, 3 \rangle$.

Example: Find scalar parametric equations for the line through the origin parallel to this line.

Point on line: $(0, 0, 0)$
Vector in direction of line: $\langle 1, 2, 3 \rangle$.
Vector equation: $\vec{r} = \langle 0, 0, 0 \rangle + t \langle 1, 2, 3 \rangle$.
Scalar equations: $x = t \quad y = 2t \quad z = 3t$.

Note: You can also read in the textbook about symmetric equations for a line.

Definition: A *vector equation* for the plane perpendicular to the vector $\vec{n} = \langle a, b, c\rangle$ containing the point (x_0, y_0, z_0) with position vector $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$ is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

We can turn this into a scalar equation (a linear equation), sometimes called an implicit equation by WeBWorK, as follows:

$$\langle a, b, c\rangle \cdot (\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle) = 0$$

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Note: From a linear equation $ax + by + cz = d$ for a plane, you can read off the normal vector $\vec{n} = \langle a, b, c\rangle$.

Example: Find an equation for the plane containing the point $(1, 2, 5)$ parallel to the plane with equation

$$x - 2y + z = 5.$$

Vector normal to plane: $\vec{n} = \langle 1, -2, 1\rangle$.
Point on plane: $(x_0, y_0, z_0) = (1, 2, 5)$.
Linear equation: $x - 2y + z = 1 - 2(2) + 5 = 2$.
Vector equation: $\langle x - 1, y - 2, z - 5\rangle \cdot \langle 1, -2, 1\rangle = \langle 0, 0, 0\rangle$.
Or: $(\vec{r} - \langle 1, 2, 5\rangle) \cdot \langle 1, -2, 1\rangle = \vec{0}$.

Example: Find a linear equation for the plane containing the points $(1,1,1)$, $(1,2,3)$, and $(-1,-1,1)$.

Vectors parallel to plane: $\langle 1,2,3\rangle - \langle 1,1,1\rangle = \langle 0,1,2\rangle$, $\langle -1,-1,1,\rangle - \langle 1,1,1\rangle = \langle -2,-2,0\rangle$.
Vector normal to plane: $\langle 0,1,2\rangle \times \langle -2,-2,0\rangle = \langle 4,-4,2\rangle$.
Point on plane: $\langle 1,1,1\rangle$
Equation: $4x - 4y + 2z = 4(1) - 4(1) + 2(1) = 2$, or $\boxed{2x - 2y + z = 1}$.
To check: See whether all 3 points satisfy the equation.

Definition: A vector parametric equation for the plane containing the point with position vector $\vec{r}_0$ and parallel to both vectors $\vec{v}$ and $\vec{w}$ (which are not parallel to each other) is

$$\vec{r} = \vec{r}_0 + t\vec{v} + s\vec{w}.$$

We will not use vector parametric equations for planes until later in the course, when we talk about surfaces in general. However, understanding vector parametric equations for planes is a good warm-up for that part of the course.

Definition: Planes are called parallel if they have parallel normal vectors.
The angle between two planes is the angle between their normal vectors.

Example: Find the distance from the point $(1, 2, 3)$ to the plane with equation

$$x + y + z = 3.$$

Example: Does the line through the points $(7, 9, 3)$ and $(-2, -3, 0)$ intersect the line through the points $(2, 2, 3)$ and $(0, 0, -1)$?

Example: Find the distance between the skew lines (lines that are not parallel but do not meet) $\vec{r} = \langle -1, -2, 1 \rangle + t \langle 1, -1, 0 \rangle$ and $\vec{r} = \langle 1, 0, 1 \rangle + t \langle 1, 1, -1 \rangle$.

To do this, find a vector $\vec{n}$ that is perpendicular to both lines. Then take any points P on the first line and Q on the second line, and take the absolute value of the scalar projection of $\overrightarrow{PQ}$ on the vector $\vec{n}$.

Explain why this works. (Hint: Think of writing $\overrightarrow{PQ}$ as the sum of three components, one parallel to the first line, one perpendicular to both lines, and one parallel to the second line. Draw a picture, with these vectors position head-to-tail making a path starting at P and ending at Q. Explain why the length of the second leg of the path is the distance between the lines.)

Do not try to memorize the various formulas for the distances between a point and a line, between a point and a plane, between two lines, between a line and a plane, between two planes.

Example: Find a linear equation for the plane through the origin that is parallel to both the lines $\vec{r} = \langle -1, -2, 1 \rangle + t \langle 1, -1, 0 \rangle$ and $\vec{r} = \langle 1, 0, 1 \rangle + t \langle 1, 1, -1 \rangle$.

Example: Find the distance between the parallel planes

$$x + 2y - z = 20$$

$$2x + 4y - 2z = 20.$$