Math 11
Fall 2016
Section 1
Monday, September 19, 2016

First, some important points from the last class:

Definition: A vector parametric equation for the line parallel to vector $\vec{v} = \langle x_v, y_v, z_v \rangle$ passing through the point (x_0, y_0, z_0) with position vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is

$$\vec{r} = \vec{r_0} + t\vec{v}, \text{ or}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle x_v, y_v, z_v \rangle .$$

Scalar parametric equations for this line are

$$x = x_0 + tx_v \qquad y = y_0 + ty_v \qquad z = z_0 + tz_v.$$

Definition: A vector equation for the plane perpendicular to the vector $\vec{n} = \langle a, b, c \rangle$ containing the point (x_0, y_0, z_0) with position vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

The scalar (linear) equation is:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Note: From a linear equation $ax + by + cz = d$ for a plane, you can read off the normal vector $\vec{n} = \langle a, b, c \rangle$.

Definition: Planes are called parallel if they have parallel normal vectors.

The angle between two planes is the angle between their normal vectors.

Last time we saw that if an object starts at point (a, b, c) with position vector $\vec{r}_0$ and moves with constant velocity $\vec{v} = \langle v_x,\ v_y,\ v_z\rangle$ for t seconds, then its final position is given by $(a + v_xt,\ b + v_yt,\ c + v_zt)$.

We can express this by a function that takes the elapsed time t to the position vector after t seconds:

$$\vec{f}(t) = \langle a + v_xt,\ b + v_yt,\ c + v_zt\rangle = \langle a, b, c\rangle + t\,\langle v_x, v_y, v_z\rangle\,,$$

$$\underbrace{\vec{f}(t)}_{\text{position}} = \underbrace{\vec{r}_0}_{\text{initial position}} + \underbrace{\underbrace{t}_{\text{time}}\ \underbrace{\vec{v}}_{\text{velocity}}}_{\text{displacement}}\,,$$

where t represents time in seconds, $t = 0$ is the starting time, and $\vec{f}(t)$ is the object's position vector at time t. The domain of $\vec{f}$ is the real number line $\mathbb{R}$ and the range of $\vec{f}$ is a line contained in three-dimensional space $\mathbb{R}^3$

Suppose another object is traveling counterclockwise around the unit circle $x^2 + y^2 = 1$ in the plane $\mathbb{R}^2$. At time $t = 0$ it is at the point $(1, 0)$, and it travels at constant speed, making one complete trip around the circle in 2π units of time.

1. What is the angle between the object's position vector and the positive x-axis when $t = \dfrac{\pi}{2}$? What is the object's position at that time?

   $\dfrac{\pi}{2}$ and $(0, 1)$, respectively. (When $t = 2\pi$ it has completed one trip around the circle, through an angle of 2π, so when $t = \dfrac{\pi}{2}$ it has gone a quarter of the way around.)

2. At what time $t > 0$ is the angle between the object's position vector and the positive x-axis first equal to $\dfrac{4\pi}{3}$?

   $t = \dfrac{4\pi}{3}$

3. What is the angle $\theta(t)$ between the object's position vector and the positive x-axis at time t?

   $\theta(t) = t$.

4. What is the object's position at time t?

   $(\cos(t), \sin(t))$.

In this case, too, we can write the object's position as a function of time:

$$\vec{f}(t) =$$

$$\vec{f}(t) = \langle \cos(t), \sin(t)\rangle\,.$$

A *vector valued function* $\vec{r}(t)$ is a function that takes a real number t to a vector $\vec{r}(t)$. If the range consists of vectors in $\mathbb{R}^3$, for example, we write $\vec{r}:\mathbb{R}\to\mathbb{R}^3$. We may say "$\vec{r}$ maps $\mathbb{R}$ to $\mathbb{R}^3$."

$$\underbrace{\vec{r}}_{\text{function}} \ : \ \underbrace{\mathbb{R}}_{\text{contains domain}} \ \to \ \underbrace{\mathbb{R}^3}_{\text{contains range}}$$

Example: If $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$, then $\vec{r}:\mathbb{R}\to\mathbb{R}^2$.
The domain of $\vec{r}$ is $\mathbb{R}$ and the range of $\vec{r}$ is the unit circle in $\mathbb{R}^2$

Definition: If a curve γ is the range of a vector function $\vec{r}$, we say that $\vec{r}$ parametrizes γ, or is a parametrization of γ. (Think of t in $\vec{r}(t)$ as a parameter — different values of t give different points on γ. You can also think $\vec{r}(t)$ is the position at time t of a point moving along γ.)
For example, $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ parametrizes a line, and $\vec{r}(t) = \langle \cos(t),\, \sin(t)\rangle$, for $0 \le t \le 2\pi$, parametrizes the unit circle in $\mathbb{R}^2$.

Sometimes we speak of *oriented* curves, which are curves with a chosen direction of motion. For example, the unit circle gives rise to two oriented curves, one oriented clockwise and one oriented counterclockwise. The function

$$\vec{r}(t) = \langle \cos(t),\, \sin(t)\rangle \qquad 0 \le t \le 2\pi$$

parametrizes the unit circle with counterclockwise orientation. The function

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \qquad 0 \le t \le 1$$

parametrizes the line segment from $\vec{r}_0$ to $\vec{r}_1$, oriented in that direction.

Example: Give a parametrization of the curve in $\mathbb{R}^3$ that is the intersection of the surface $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$.

The projection on the xy-plane is the unit circle $x^2 + y^2 = 1$, so we can set $x = \cos(t)$ and $y = \sin(t)$. Then $z = x^2 - y^2 = \cos^2(t) - \sin^2(t) = \cos(2t)$.

$$\vec{f}(t) = \langle \cos(t),\, \sin(t),\, \cos(2t)\rangle\,.$$

We can take $0 \le t \le 2\pi$, if we want to trace out the entire curve without retracing any part.

We prefer our parametrizations to be continuous.

You may recall that for $f : \mathbb{R} \to \mathbb{R}$ we say

$\lim_{x\to a} f(x) = L$ means for every $\varepsilon > 0$ [desired output accuracy] there is a $\delta > 0$ [required input accuracy] such that, for every x,

$$\underbrace{|x-a| < \delta \ \&\ x \neq a}_{\text{within input accuracy of } a\text{, but not } a} \quad \Longrightarrow \quad \underbrace{|f(x) - L| < \varepsilon}_{\text{within output accuracy of } L} \quad .$$

If $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, we say something very similar

$\lim_{t\to a} \vec{r}(t) = \vec{L}$ means for every $\varepsilon > 0$ [desired output accuracy] there is a $\delta > 0$ [required input accuracy] such that, for every t,

$$\underbrace{|t-a| < \delta \ \&\ t \neq a}_{\text{within input accuracy of } a\text{, but not } a} \quad \Longrightarrow \quad \underbrace{\overbrace{|\vec{r}(t) - \vec{L}|}^{\text{distance between } \vec{r}(t) \text{ and } \vec{L}} \qquad < \varepsilon}_{\text{within output accuracy of } \vec{L}}.$$

In practice, we don't often use this formal definition of limit to compute limits, although we may use it to prove things.

Theorem: If

$$\vec{r}(t) = \langle r_x(t), r_y(t), r_z(t) \rangle \, ,$$

then

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} r_x(t),\ \lim_{t\to a} r_y(t),\ \lim_{t\to a} r_z(t) \right\rangle .$$

Example:

$$\lim_{t\to 0} \left\langle \frac{\sin(t)}{t},\ t^2 \ln(t^2) \right\rangle =$$

$$\left\langle \lim_{t\to 0} \frac{\sin(t)}{t},\ \lim_{t\to 0} t^2 \ln(t^2) \right\rangle =$$

(using l'Hôpital's rule)

$$\left\langle \lim_{t\to 0} \frac{\cos(t)}{1},\ \lim_{t\to 0} \frac{\ln(t^2)}{\frac{1}{t^2}} \right\rangle =$$

$$\left\langle 1,\ \lim_{t\to 0} \frac{2t\frac{1}{t^2}}{\frac{-2}{t^3}} \right\rangle =$$

$$\langle 1, 0 \rangle \, .$$

Definition: A vector function $\vec{r}(t)$ is continuous at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

Example: Give parametrizations of the following curves:

1. The intersection of the paraboloid $z = x^2 + y^2$ and the plane $x = 1$.

2. The ellipse $x^2 + 4y^2 = 4$ in $\mathbb{R}^2$.

Example: Sketch and/or describe the curves parametrized by the following functions:

1. $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

2. $\vec{r}(t) = \langle t, t, t^2 \rangle$.

We determined that if an object travels with constant velocity $\vec{v}$ for time period t, its displacement is

$$\vec{d} = t\vec{v}.$$

We can use this the other way: If an object travels with constant velocity for time period t and its displacement is $\vec{d}$, then its velocity is

$$\vec{v} = \frac{1}{t}\,\vec{d}.$$

If the velocity is not constant, we say $\frac{1}{t}\,\vec{d}$ is the average velocity.

Example: An object travels around the unit circle in the xy-plane, and its position at time t is the point $(\cos t, \sin t)$ (with time and distance measured in your favorite units).

The object's displacement between times t and $t + \Delta t$ is
$\langle \cos(t+\Delta t), \sin(t+\Delta t)\rangle - \langle \cos(t), \sin(t)\rangle =$
$\boxed{\langle \cos(t+\Delta t) - \cos(t), \sin(t+\Delta t) - \sin(t)\rangle}$

If Δt is small enough, it is not a bad approximation to suppose that between times t and Δt, the object is traveling at constant velocity.

The object's velocity between times t and Δt is approximately
$\frac{1}{\Delta t}\langle \cos(t+\Delta t) - \cos(t), \sin(t+\Delta t) - \sin(t)\rangle =$
$\boxed{\left\langle \frac{\cos(t+\Delta t) - \cos(t)}{\Delta t}, \frac{\sin(t+\Delta t) - \sin(t)}{\Delta t} \right\rangle}$

The object's instantaneous velocity at time t is (letting $\Delta t \to 0$)

$$\lim_{\Delta t \to 0} \left\langle \frac{\cos(t+\Delta t) - \cos(t)}{\Delta t}, \frac{\sin(t+\Delta t) - \sin(t)}{\Delta t} \right\rangle =$$

$$\left\langle \lim_{\Delta t \to 0} \frac{\cos(t+\Delta t) - \cos(t)}{\Delta t},\ \lim_{\Delta t \to 0} \frac{\sin(t+\Delta t) - \sin(t)}{\Delta t} \right\rangle = \langle -\sin(t),\ \cos(t)\rangle\,.$$

Definition: The derivative of a vector function is defined by

$$\frac{d}{dt}\vec{r}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\vec{r}(t + \Delta t) - \vec{r}(t) \right).$$

Theorem: If

$$\vec{r}(t) = \langle r_x(t), r_y(t), r_z(t) \rangle ,$$

then

$$\frac{d}{dt}\vec{r}(t) = \left\langle \frac{d}{dt} r_x(t),\ \frac{d}{dt} r_y(t),\ \frac{d}{dt} r_z(t) \right\rangle .$$

Example:

$$\frac{d}{dt} \langle \cos(t),\ \sin(t) \rangle = \langle -\sin(t),\ \cos(t) \rangle .$$

If $\vec{r}(t)$ denotes the position of a moving object at time t, then the derivative $\vec{r}\,'(t)$ denotes the velocity of that object, its magnitude $|\vec{r}\,'(t)|$ is the object's speed, and the unit vector $\vec{T}(t)$ in the direction of $\vec{r}\,'(t)$ gives the direction of motion.

The unit vector $\vec{T}(t)$ is tangent to the object's path, and is called the unit tangent vector.

Example: An object moves in the plane with position function

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle .$$

Find a vector parametric equation for the line tangent to the object's path at the point $\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

$\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \vec{r}\left(\frac{\pi}{3}\right)$ is the object's position at time $t = \frac{\pi}{3}$. This is a point on the tangent line.

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$ is the object's velocity at time t.

$\vec{r}\,'\left(\frac{\pi}{3}\right) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ is the object's velocity at time $t = \frac{\pi}{3}$. Since velocity points in the direction of motion, this is a vector in the direction of the tangent line.

$\vec{r} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle + t \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ is an equation for the tangent line.

The function $\vec{\ell}(t) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle + t \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ parametrizes the tangent line.

Notice, the function $\vec{\ell}(t)$ represents the motion of an object traveling along the tangent line at the same velocity our original object has at time $t = \frac{\pi}{3}$, starting at time $t = 0$ at the position of our original object at time $t = \frac{\pi}{3}$.

If we want the object with position function ℓ to be at the same point as our original object at time $t = \frac{\pi}{3}$, we should adjust its clock. We can do this by replacing t with $t - \frac{\pi}{3}$, to get a new position function

$\vec{f}(t) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle + \left(t - \frac{\pi}{3}\right) \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$.

You can check this function has the same value and derivative as $\vec{r}(t)$ when $t = \frac{\pi}{3}$.

Theorem: If all functions mentioned are differentiable, then

$$\frac{d}{dt}(a\vec{r}(t) + b\vec{p}(t)) = a\vec{r}\,'(t) + b\vec{p}\,'(t)$$

$$\frac{d}{dt}(\vec{r}(f(t)) = f'(t)\vec{r}\,'(f(t))$$

$$\frac{d}{dt}\left(f(t)\vec{r}(t)\right) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{p}(t)\right) = \vec{r}\,'(t) \cdot \vec{p}(t) + \vec{r}(t) \cdot \vec{p}\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t) \times \vec{p}(t)\right) = \vec{r}\,'(t) \times \vec{p}(t) + \vec{r}(t) \times \vec{p}\,'(t)$$

Theorem: If $\vec{r}(t)$ is differentiable, then $|\vec{r}(t)|$ is constant if and only if $\vec{r}(t) \perp \vec{r}\,'(t)$ for all t.

Proof: We know $|\vec{r}(t)|$ is constant if and only if $|\vec{r}(t)|^2$ is constant, and $|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$. A function is constant if and only if its derivative is always zero, so $|\vec{r}(t)|^2$ is constant if and only if

$$0 = \frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 2\vec{r}(t) \cdot \vec{r}\,'(t).$$

We also know that $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$ if and only if $\vec{r}(t) \perp \vec{r}\,'(t)$.

Putting all this together, $|\vec{r}(t)|$ is constant if and only if $\vec{r}(t) \perp \vec{r}\,'(t)$ for all t.

Definition: If a curve γ is the range of a vector function $\vec{r}$, we say that $\vec{r}$ parametrizes γ, or is a parametrization of γ. (Think of t in $\vec{r}(t)$ as a parameter — different values of t give different points on γ.)

If $\vec{r}\,'(t)$ is defined and nonzero for every t, then $\vec{r}$ is a regular parametrization, or smooth parametrization, of γ. A curve with a regular parametrization is called a smooth curve.

Definition:

$$\int \langle r_x(t), r_y(t), r_z(t)\rangle \; dt = \left\langle \int r_x(t)\,dt,\; \int r_y(t)\,dt,\; \int r_z(t)\,dt \right\rangle$$

$$\int_a^b \langle r_x(t), r_y(t), r_z(t)\rangle \; dt = \left\langle \int_a^b r_x(t)\,dt,\; \int_a^b r_y(t)\,dt,\; \int_a^b r_z(t)\,dt \right\rangle$$

Theorem

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a).$$

If $\vec{r}(t)$ is the position of a moving object at time t then this is:
the net displacement between times $t = a$ and $t = b$.

Example: The position function of a moving object is

$$\vec{r}(t) = \langle 1, 3\cos(t), 3\sin(t) \rangle .$$

1. Describe the object's path as completely as possible.

2. Find the object's velocity in general and at time $t = \dfrac{\pi}{2}$.

3. Find an equation for the line tangent to the object's path at the point $\langle 1, 0, 3 \rangle$.

4. Find $\displaystyle\int_0^{\pi} \vec{r}\,'(t)\,dt$. (What does this represent?)

5. Find the distance between the object's position at $t = 0$ and the object's position at $t = \pi$. (How is this related to your answer to part (4)?)

6. Using part (1), find the length of the curve the object travels along between time $t = 0$ and time $t = \pi$. (Don't try to use calculus for this one. Use geometry.)

Example: If the position of a moving object at time t is $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$, what can we say about the motion of that object?

Once you have said everything else you have to say: What is the projection of the object's path on the xy-plane? What is the angle between the object's velocity vector at time t and the xy-plane? (Think about this one. It requires some cleverness.) Given that, can you make a guess about the length of the path the object travels along between $t = 0$ and $t = 2\pi$?

We will soon learn how to compute lengths of paths.

Challenge: Give parametrizations of the following curves:

1. The portion of the intersection of the sphere $x^2 + y^2 + z^2 = 4$ with the plane $x = y + 1$ for which $x \geq 0$ and $y \geq 0$.

2. The intersection of the plane $z = 1$ and the surface $z = x^4 + y^4$.

Challenge: Sketch and/or describe the curves parametrized by the following functions:

1. $\vec{r}(t) = \cos(t)\left\langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right\rangle + \sin(t)\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right\rangle$. You may notice that $\left\langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right\rangle$ are unit vectors and are perpendicular to each other.

2. The curve that lies in the cone $z = \sqrt{x^2 + y^2}$ and whose projection onto the xy-plane is parametrized by $\vec{r}(t) = \langle t\cos(t), t\sin(t) \rangle$.