Math 11
Fall 2016
Section 1
Tuesday, September 21, 2016

First, some important points from the last class:

A *vector valued function* $\vec{r}(t)$ is a function that takes a real number t to a vector $\vec{r}(t)$. If the range consists of vectors in $\mathbb{R}^3$, for example, we write $\vec{r}:\mathbb{R}\to\mathbb{R}^3$. We may say "$\vec{r}$ maps $\mathbb{R}$ to $\mathbb{R}^3$."

$$\underbrace{\vec{r}}_{\text{function}} : \underbrace{\mathbb{R}}_{\text{contains domain}} \to \underbrace{\mathbb{R}^3}_{\text{contains range}}$$

Definition: $\lim_{t\to a}\vec{r}(t)=\vec{L}$ means for every $\varepsilon>0$ [desired output accuracy] there is a $\delta>0$ [required input accuracy] such that, for every t,

$$\underbrace{|t-a|<\delta \ \&\ t\neq a}_{\text{within input accuracy but not } a} \implies \underbrace{\overbrace{|\vec{r}(t)-\vec{L}|}^{\text{distance between } \vec{r}(t) \text{ and } \vec{L}} < \varepsilon}_{\text{within output accuracy}}.$$

Theorem: If

$$\vec{r}(t)=\langle r_x(t), r_y(t), r_z(t)\rangle,$$

then

$$\lim_{t\to a}\vec{r}(t)=\left\langle \lim_{t\to a} r_x(t),\ \lim_{t\to a} r_y(t),\ \lim_{t\to a} r_z(t)\right\rangle.$$

Definition: A vector function $\vec{r}(t)$ is continuous at a if $\lim_{t\to a}\vec{r}(t)=\vec{r}(a)$.

Definition: The vector function $\vec{r}(t)$ parametrizes the curve γ if γ is the range of $\vec{r}(t)$. The curve γ is given an *orientation* (choice of direction) by the direction of motion of $\vec{r}(t)$ as t increases.

Methods to try for parametrizing curves given as intersections of surfaces:

I. Eliminate variables by using the equations of the surfaces. If, for example, you can write y and z in terms of x, then you can set $x=t$.

II. If you have an equation of the form $A^2+B^2=1$, where A and B are expressions involving two variables, set $A=\cos(t)$ and $B=\sin(t)$.

III. Use polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ for curves circling the origin in $\mathbb{R}^2$, by writing r in terms of θ, and setting $\theta=t$.

Example: Parametrize the curve $x^4 + y^4 = 1$ in $\mathbb{R}^2$.

We set

$$x = r\cos\theta \qquad y = r\sin\theta$$

so our equation becomes

$$\begin{aligned} r^4\cos^4\theta + r^4\sin^4\theta &= 1; \\ r^4 &= (\cos^4\theta + \sin^4\theta)^{-1}; \\ r &= (\cos^4\theta + \sin^4\theta)^{-\frac{1}{4}}. \end{aligned}$$

Now we have

$$x = r\cos\theta = (\cos^4\theta + \sin^4\theta)^{-\frac{1}{4}}\cos\theta \qquad y = r\sin\theta = (\cos^4\theta + \sin^4\theta)^{-\frac{1}{4}}\sin\theta,$$

and we can set $t = \theta$ to get

$$\vec{r}(t) = \left\langle (\cos^4 t + \sin^4 t)^{-\frac{1}{4}}\cos t,\ (\cos^4 t + \sin^4 t)^{-\frac{1}{4}}\sin t \right\rangle \qquad 0 \le t \le 2\pi.$$

Definition: The derivative of a vector function is defined by

$$\frac{d}{dt}\vec{r}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\vec{r}(t + \Delta t) - \vec{r}(t) \right).$$

Theorem: If

$$\vec{r}(t) = \langle r_x(t), r_y(t), r_z(t) \rangle,$$

then

$$\frac{d}{dt}\vec{r}(t) = \left\langle \frac{d}{dt} r_x(t),\ \frac{d}{dt} r_y(t),\ \frac{d}{dt} r_z(t) \right\rangle.$$

Theorem: If all functions mentioned are differentiable, then

$$\frac{d}{dt}(a\vec{r}(t) + b\vec{p}(t)) = a\vec{r}\,'(t) + b\vec{p}\,'(t)$$

$$\frac{d}{dt}(\vec{r}(f(t)) = f'(t)\vec{r}\,'(f(t))$$

$$\frac{d}{dt}\left(f(t)\vec{r}(t)\right) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{p}(t)\right) = \vec{r}\,'(t) \cdot \vec{p}(t) + \vec{r}(t) \cdot \vec{p}\,'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t) \times \vec{p}(t)\right) = \vec{r}\,'(t) \times \vec{p}(t) + \vec{r}(t) \times \vec{p}\,'(t)$$

Theorem: If $\vec{r}(t)$ is differentiable, then $|\vec{r}(t)|$ is constant if and only if $\vec{r}(t) \perp \vec{r}\,'(t)$ for all t.

Definition: If $\vec{r}\,'(t)$ is defined and nonzero for every t, then $\vec{r}$ is a regular parametrization, or smooth parametrization, of its image γ. A curve with a regular parametrization is called a smooth curve.

Definition:

$$\int \langle r_x(t), r_y(t), r_z(t) \rangle \, dt = \left\langle \int r_x(t)\, dt,\ \int r_y(t)\, dt,\ \int r_z(t)\, dt \right\rangle$$

$$\int_a^b \langle r_x(t), r_y(t), r_z(t) \rangle \, dt = \left\langle \int_a^b r_x(t)\, dt,\ \int_a^b r_y(t)\, dt,\ \int_a^b r_z(t)\, dt \right\rangle$$

Theorem:

$$\int_a^b \vec{r}\,'(t)\, dt = \vec{r}(b) - \vec{r}(a).$$

If $\vec{r}(t)$ is the position of a moving object at time t then this is the net displacement between times $t = a$ and $t = b$.

Note: If $\vec{r}(t)$ is a vector-valued function, then for t near t_0, we can approximate $\vec{r}(t)$ by

$$\vec{L}(t) = \vec{r}(t_0) + (t - t_0)\,\vec{r}\,'(t_0).$$

This is exactly like the tangent line approximation for functions from $\mathbb{R}$ to $\mathbb{R}$.

We can rewrite this a bit:

$$\vec{L}(t) = \underbrace{\vec{r}(t_0) - t_0\vec{r}\,'(t_0)}_{\text{constant}} + t\,\underbrace{\vec{r}\,'(t_0)}_{\text{constant}}.$$

If $\vec{r}$ parametrizes the curve γ, then $\vec{L}$ parametrizes the tangent line to γ at the point $\vec{r}(t_0)$. (Unless $\vec{r}\,'(t_0) = \vec{0}$, in which case $\vec{L}$ is a constant function with value $\vec{r}(t_0)$.)

If $\vec{r}$ is a position function of a moving object, then $\vec{L}$ is the position function of another object moving with constant velocity, that has the same position and velocity as our original object at time t_0.

Example: Find a vector parametric equation for the tangent line to the curve γ parametrized by $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$.

This point is $\vec{r}\left(\frac{\pi}{4}\right)$, so we can set $t_0 = \frac{\pi}{4}$, and parametrize the tangent line by

$$\vec{L}(t) = \vec{r}(t_0) + (t - t_0)\,\vec{r}\,'(t_0) = \langle \cos(t_0), \sin(t_0), t_0\rangle + (t - t_0)\,\langle -\sin(t_0), \cos(t_0), 1\rangle =$$

$$\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right\rangle + \left(t - \frac{\pi}{4}\right)\left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right\rangle.$$

A vector parametric equation for this line is

$$\langle x, y, z\rangle = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right\rangle + \left(t - \frac{\pi}{4}\right)\left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right\rangle =$$

$$\left\langle \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{4}\right), \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right), 0\right\rangle + t\left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right\rangle.$$

1. Suppose an object is traveling with variable velocity $\vec{v}(t)$ for a period of time of length Δt, and t_i is some particular time in that period. If Δt is small enough, then over that period of time

   (a) The object's displacement is approximately $\boxed{\vec{v}(t_i)\,\Delta t}$;

   (b) The distance the object travels is approximately $\boxed{|\vec{v}(t_i)|\,\Delta t}$.

2. If the object travels between times $t = a$ and $t = b$, we break that time period up into n-many small periods of time of length Δt, and for $i = 1, \ldots, n$ we choose time t_i in the i^{th} time period, then

   (a) The distance the object travels during the i^{th} time period is approximately $\boxed{|\vec{v}(t_i)|\,\Delta t}$;

   (b) The total distance the object travels is approximately $\boxed{\displaystyle\sum_{i=1}^{n} |\vec{v}(t_i)|\,\Delta t}$.

3. Taking a limit as $n \to \infty$, the total distance the object travels is

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} |\vec{v}(t_i)|\,\Delta t = \int_a^b |\vec{v}(t)|\,dt}.$$

   Note, this is not the same as net distance, which is the magnitude of the displacement. If an object travels once around the unit circle, the net distance it moves is 0, since it ends up where it started. However, the total distance it travels is the length of its path, which is 2π.

   We get displacement by integrating velocity, and distance traveled (length of path) by integrating speed.

Definition: If a curve γ has a regular (smooth) parametrization

$$\vec{r} : [a, b] \to \mathbb{R}^n$$

that does not retrace any portion of γ, then the arc length of γ is

$$L = \int_a^b |\vec{r}\,'(t)|\, dt.$$

The arc length function is the function

$$s : [a, b] \to [0, L]$$

that takes t to the arc length of the portion of γ between $\vec{r}(a)$ and $\vec{r}(t)$:

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du.$$

The parametrization of γ by arc length is the function that takes a number s to the point on γ that is a distance of s units along γ from the starting point $\vec{r}(a)$.

We sometimes denote speed by $\dfrac{ds}{dt}$, the rate of change of distance with respect to time.

Example: Let γ be the circle of radius a centered at the origin in $\mathbb{R}^2$.

To find the arc length of γ we need a parametrization:

$$\vec{r}(t) = \langle a\cos t, a\sin t\rangle \qquad 0 \leq 2\pi;$$

$$\vec{r}\,'(t) = \langle -a\sin t, a\cos t\rangle\,;$$

$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a;$$

$$\text{arc length } = \int_0^{2\pi} \frac{ds}{dt}\, dt = \int_0^{2\pi} a\, dt = 2\pi a.$$

To find a parametrization by arc length, we find the arc length s between times 0 and t:

$$s = \int_0^t |\vec{r}\,'(u)|\; du = \int_0^t a\, du = au\Big|_{u=0}^{u=t} = at.$$

Rewrite t in terms of s:

$$t = \frac{s}{a}.$$

Plug into the given parametrization:

$$\vec{r} = \langle a\cos t, a\sin t\rangle = \vec{r}(t(s)) = \left\langle a\cos\left(\frac{s}{a}\right), a\sin\left(\frac{s}{a}\right)\right\rangle.$$

Definition: If $\vec{r}(t)$ is the position of a moving object at time t, its acceleration at time t is

$$\vec{a}(t) = \vec{r}\,''(t).$$

Example: The acceleration of gravity near the surface of the earth is $\langle 0, 0, -g \rangle$, where g is a constant. At time $t = 0$ a projectile is launched from the origin with initial velocity $\langle 1, 2, 3 \rangle$. Find the time at which it hits the ground (the xy-plane).

$$\vec{a}(t) = \langle 0, 0, -g \rangle .$$

Integrate, not forgetting the constant of integration:

$$\vec{v}(t) = \langle C_1, C_2, -gt + C_3 \rangle = \langle 0, 0, -gt \rangle + \langle C_1, C_2, C_3 \rangle .$$

Use initial conditions to solve for the constant of integration:

$$\vec{v}(0) = \langle C_1, C_2, C_3 \rangle = \langle 1, 2, 3 \rangle ;$$

$$\vec{v}(t) = \langle 1, 2, -gt + 3 \rangle ;$$

$$\vec{r}(t) = \left\langle t, 2t, -\frac{gt^2}{2} + 3t \right\rangle + \langle D_1, D_2, D_3 \rangle ;$$

$$\vec{r}(0) = \langle D_1, D_2, D_3 \rangle = \langle 0, 0, 0 \rangle ;$$

$$\vec{r}(t) = \left\langle t, 2t, -\frac{gt^2}{2} + 3t \right\rangle .$$

The object is at the xy-plane when its z-coordinate is 0, or

$$0 = -\frac{gt^2}{2} + 3t = \left(-\frac{gt}{2} + 3 \right) t = 0;$$

This happens initially (when $t = 0$) and when

$$\frac{gt}{2} = 3;$$

$$t = \frac{6}{g}.$$

Analyzing acceleration: (This is cultural enrichment. You can read about it in the textbook, and if you plan to take physics, you should.)

Given the position $\vec{r}$ of an moving object as a function of the time t, we can compute:

$$\vec{v} = \frac{d\vec{r}}{dt} = \text{ velocity} \qquad \frac{ds}{dt} = |\vec{v}| = \text{ speed} \qquad \vec{v} = |\vec{v}|\,\vec{T} = \frac{ds}{dt}\vec{T}$$

$$\vec{T} = \frac{1}{\frac{ds}{dt}}\vec{v} = \text{ unit tangent vector } = \text{ unit vector in direction of motion}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \text{ acceleration}$$

By an earlier theorem, because $|\vec{T}|$ is constant, we have $\frac{d\vec{T}}{dt} \perp \vec{T}$. We define the *unit normal vector* $\vec{N}$ to be the unit vector in this direction,

$$\vec{N} = \frac{1}{\left\|\frac{d\vec{T}}{dt}\right\|}\frac{d\vec{T}}{dt} \qquad \frac{d\vec{T}}{dt} = \left\|\frac{d\vec{T}}{dt}\right\|\vec{N}$$

Using the product rule, we can write

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\vec{T}\right) = \left(\frac{d}{dt}\frac{ds}{dt}\right)\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt} = \underbrace{\left(\frac{d^2s}{dt^2}\right)}_{a_{\mathbf{T}}}\vec{T} + \underbrace{\frac{ds}{dt}\left|\frac{d\vec{T}}{dt}\right|}_{a_{\mathbf{N}}}\vec{N}$$

The acceleration $\vec{a}$ is expressed as the sum of two parts, one in the direction of motion, and one normal (perpendicular) to the direction of motion.

The tangential component of the acceleration is $a_{\mathbf{T}} = \frac{d^2s}{dt^2}$, the second derivative of distance with respect to time. We may call this the linear acceleration. It is the scalar projection of acceleration in the direction of motion.

The normal component of the acceleration is $a_{\mathbf{N}} = \frac{ds}{dt}\left|\frac{d\vec{T}}{dt}\right|$. It turns out that this can also be written as $\left(\frac{ds}{dt}\right)^2\kappa$, where κ is the *curvature* of γ.

The tangential component $a_{\mathbf{T}}$ reflects changing speed, and the normal component $a_{\mathbf{N}}$ reflects changing direction. They are associated with the components of the force acting on the object in the direction of motion and normal to the direction of motion, which we may call these the linear force and the centripetal force.

Example: The position of a moving object at time t is given by $\vec{r}(t) = \langle t, \sin(t), \cos(t) \rangle$ for $0 \leq t \leq 2\pi$.

Find its velocity and acceleration.

$$\vec{v} = \langle 1, \cos(t), -\sin(t) \rangle \qquad \vec{a} = \langle 0, -\sin(t), -\cos(t) \rangle$$

Find the total distance it travels, and its arc length function.

$$L = \int_0^{2\pi} |\langle 1, \cos(t), -\sin(t) \rangle| \; dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\,\pi.$$

$$s = \int_0^{t} |\langle 1, \cos(u), -\sin(u) \rangle| \; du = \int_0^{t} \sqrt{2}\, du = \sqrt{2}\, t;$$

Find the unit tangent vector to the object's path when $t = \pi$.

$$\vec{T}(t) = \frac{1}{|\vec{v}(t)|}\vec{v}(t) = \frac{1}{\sqrt{2}}\vec{v}(t) = \frac{1}{\sqrt{2}} \langle 1, \cos(t), -\sin(t) \rangle);$$

$$\vec{T}(\pi) = \frac{1}{\sqrt{2}} \langle 1, -1, 0 \rangle .$$

Example: Consider the curve γ in $\mathbb{R}^2$ defined by

$$\vec{r}(t) = \left\langle 2\cos(t^2), 2\sin(t^2) \right\rangle$$

for $0 \leq t \leq 1$. Find the length of γ.

Arc length:

$$\int_0^1 |\vec{r}\,'(t)|\, dt = \int_0^1 4t\, dt = \boxed{2}.$$

Example: Suppose

$$f : [a, b] \to [0, 2\pi]$$

is any differentiable function with positive derivative at every point, $f(a) = 0$, and $f(b) = 2\pi$. Parametrize the unit circle by

$$\vec{r}(t) = \langle \cos(f(t)), \sin(f(t)) \rangle \qquad a \leq t \leq b.$$

Use this parametrization to compute the arc length of the unit circle.

Of course, your answer should be 2π. The point is to demonstrate why it does not matter which parametrization you choose.

$$\vec{v} = \langle -f'(t)\sin(f(t)), f'(t)\cos(f(t)) \rangle$$

$$\frac{ds}{dt} = |\langle -f'(t)\sin(f(t)), f'(t)\cos(f(t)) \rangle| = f'(t)$$

$$L = \int_a^b f'(t)\, dt = f(b) - f(a) = 2\pi.$$

Example: Parametrize the intersection of the elliptical cone $z^2 = x^2 + 4y^2$ with the plane $z = 2$.

$$\vec{r}(t) = \langle 2\cos(t), \sin(t), 2\rangle \qquad 0 \le t \le 2\pi$$

Write down an integral representing the arc length of this curve. (Do not try to evaluate this integral.)

$$\int_0^{2\pi} \sqrt{4\cos^2(t) + \sin^2(t) + 4}\, dt$$

If an object travels along the curve with position function given by the parametrization you chose, find the object's acceleration at the points $(2, 0, 2)$ and $(0, 1, 2)$.

$$\vec{v} = \langle -2\sin(t), \cos(t), 0\rangle \qquad \vec{a} = \langle -2\cos(t), -\sin(t), 0\rangle$$

At $(2, 0, 2)$, we have $t = 0$, so

$$\vec{a} = \langle -2, 0, 0\rangle \,.$$

At $(0, 1, 2)$, we have $t = \dfrac{\pi}{2}$, so

$$\vec{a} = \langle 0, -1, 0\rangle \,.$$

Example: The curve γ is the intersection of the surface $z = x^2 - y^2$ with the plane $x = 3$. Find a vector parametric equation for the tangent line to γ at the point $(3, 1, 8)$.

Find a definite integral giving the arc length of the portion of γ for $-1 \leq y \leq 1$. You do not have to evaluate this integral.