Math 11
Fall 2016
Section 1
Friday, September 23, 2016

First, some important points from the last class:

Definition: If a curve γ has a regular parametrization $\vec{r} : [a, b] \to \mathbb{R}^n$ that does not retrace any portion of γ, then the arc length of γ is

$$L = \int_a^b |\vec{r}\,'(t)|\, dt.$$

The arc length function is the function that takes t to the arc length of the portion of γ between $\vec{r}(a)$ and $\vec{r}(t)$:

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du.$$

The parametrization of γ by arc length is the function that takes a number s to the point on γ that is a distance of s units along γ from the starting point $\vec{r}(a)$.

Compute this by using the arc length function (expressing s as a function of t) to instead express t as a function of s, say $t = f(s)$, then rewriting the expression $\vec{r}(t)$ by rewriting t in terms of s, that is, by setting $\vec{p}(s) = \vec{r}(f(s))$.

Theorem: The arc length of a curve can be computed using any regular parametrization, and the answer will be the same.

We say arc length does not depend on the parametrization.

Motion:

$\vec{r} =$ position at time t;

$\vec{v} = \dfrac{d\vec{r}}{dt} =$ velocity at time t;

$\vec{T} = \dfrac{1}{|\vec{v}|}\,\vec{v} =$ unit tangent vector (in direction of motion) at time t;

$\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2} =$ acceleration at time t.

$$\vec{v} = \int \vec{a}\, dt + \vec{C};$$

$$\vec{r} = \int \vec{v}\, dt + \vec{C}.$$

Functions of several variables:

$$f : \mathbb{R}^n \to \mathbb{R}$$

Examples:

$$f(x, y) = x^2 + 4y^2 \qquad g(x, y, z) = x^2 + y^2 - z$$

Today we look at

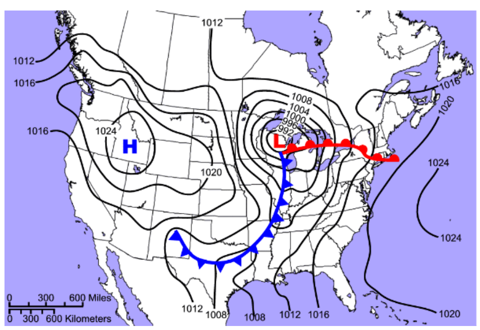
1. graphs;
2. level sets (level curves and level surfaces);
3. limits.

For $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of f is a surface in $\mathbb{R}^3$, the set of all points $(x, y, f(x, y))$.

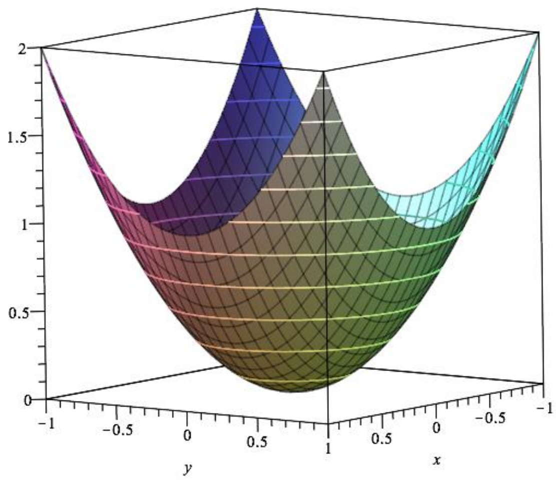
We can get an idea of the shape of the graph of f by drawing some *level curves* in $\mathbb{R}^2$, curves $f(x, y) = k$ for various values of k. If we do this for equally spaced values of k, we get something like a topographical map of the graph of f. The level curves are close together where the surface is steep.

This is called a contour map, and the level curves are also called contour lines.

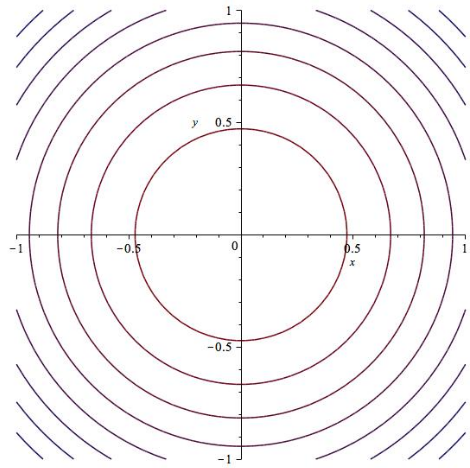
From the South Carolina State Climatology Office
http://www.dnr.sc.gov/climate/sco/Education/wxmap/wxmap.php
Isobars:



Example: $f(x,y) = x^2 + y^2$. The graph of f is a paraboloid $z = x^2 + y^2$ in $\mathbb{R}^3$. The level curves of f are circles $x^2 + y^2 = k$ of radius $\sqrt{k}$ in $\mathbb{R}^2$.

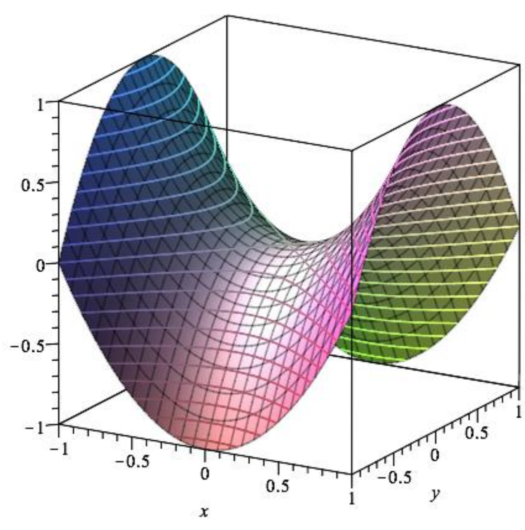


Graph of f in $\mathbb{R}^3$

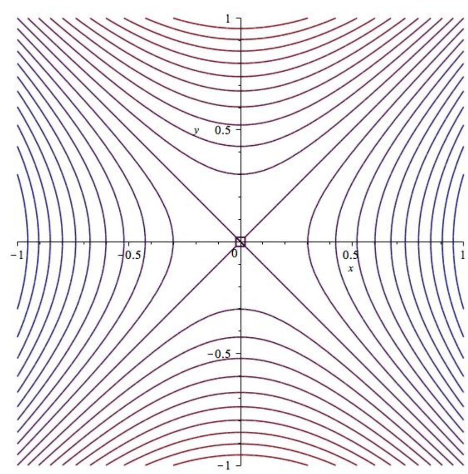


Level curves of f in $\mathbb{R}^2$

Example: $f(x,y) = x^2 - y^2$. The graph of f is a saddle $z = x^2 - y^2$ in $\mathbb{R}^3$. The level curves of f are hyperbolae $x^2 - y^2 = k$ in $\mathbb{R}^2$.



Graph of f in $\mathbb{R}^3$



Level curves of f in $\mathbb{R}^2$

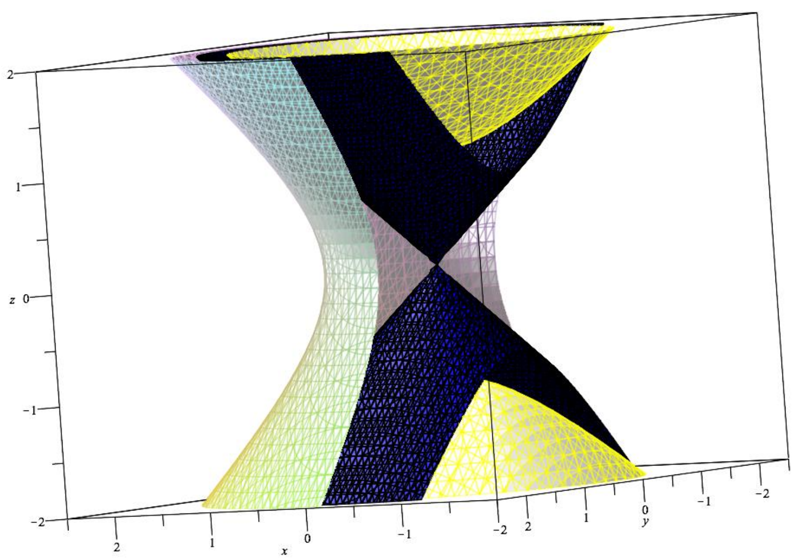
For $f : \mathbb{R}^3 \to \mathbb{R}$, the graph of f is a three-dimensional object in $\mathbb{R}^4$, the set of all points $(x, y, z, f(x, y, z))$.

We can get an idea of the shape of the graph of f by drawing some *level surfaces* in $\mathbb{R}^3$, surfaces $f(x, y, z) = k$ for various values of k. If we do this for equally spaced values of k, we get a kind of three-dimensional analogue of a topographical map.

If $f(x, y, z)$ represents some physical quantity (temperature, barometric pressure, ...), the level surfaces of f (isotherms, isobars, ...) are often used to visualize the situation.

Example: $f(x, y, z) = x^2 + y^2 + z^2$. The graph of f is a three-dimensional object sitting in $\mathbb{R}^4$. The level surfaces of f are spheres $x^2 + y^2 + z^2 = k$ of radius $\sqrt{k}$ in $\mathbb{R}^3$.

Example: $f(x, y, z) = x^2 + y^2 - z^2$. The level surfaces of f are hyperboloids — hyperboloids of two sheets for $k < 0$, a double cone for $k = 0$, and hyperboloids of one sheet for $k > 0$.



Problems: Draw some level curves of the function $f(x, y) = x^2 + y$. Then try to draw the graph of f. Remember that level curves are in the plane, $\mathbb{R}^2$, but the graph of f is a surface in $\mathbb{R}^3$.

Draw some level curves of the function $f(x, y) = 4x^2 + y^2$. Then try to draw the graph of f.

Draw some level surfaces of the function $f(x, y, z) = \dfrac{x + y}{z^2 + 1}$. Try $k = 1$, $k = 0$, $k = -1$.

Limits of functions of several variables:

Our definition of limit always has the same form: For every desired output accuracy, there is a required input accuracy, such that whenever the input is within the input accuracy, then the output is within the output accuracy.

For functions $f : \mathbb{R}^3 \to \mathbb{R}$ it becomes:

Definition:

$$\lim_{(x,y,z)\to(x_0,y_0,z_0} f(x,y,z) = L$$

means for every $\varepsilon > 0$ [desired output accuracy] there is a $\delta > 0$ [required input accuracy] such that, for every (x, y, z),

$$\Bigg[\underbrace{\overbrace{|(x,y,z)-(x_0,y_0,z_0)|}^{\text{distance between }(x,y,z)\text{ and }(x_0,y_0,z_0)} < \delta \ \ \& \ \ (x,y,z) \neq (x_0,y_0,z_0)}_{\text{within input accuracy but not }(x_0,y_0,z_0)}\Bigg] \implies \underbrace{|f(x,y,z)-L| < \varepsilon}_{\text{within output accuracy}}.$$

Definition: The function $f(x, y, z)$ is continuous at (x_0, y_0, z_0) if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0).$$

The definitions for $f : \mathbb{R}^2 \to \mathbb{R}$, and for $f : \mathbb{R}^n \to \mathbb{R}$, are similar.

If $F : \mathbb{R}^m \to \mathbb{R}^n$, we use essentially the same ε-δ definition, and we can take limits coordinatewise. So if

$$F(x,y) = (F_1(x,y), F_2(x,y)),$$

then

$$\lim_{(x,y)\to(x_0,y_0)} F(x,y) = \left(\lim_{(x,y)\to(x_0,y_0)} F_1(x,y), \lim_{(x,y)\to(x_0,y_0)} F_2(x,y)\right).$$

Warning: For $f : \mathbb{R} \to \mathbb{R}$, there are two ways x can approach a, from the right and from the left. For $f : \mathbb{R}^n \to \mathbb{R}$, there are infinitely many ways $(x_1, \ldots, x_n)$ can approach $(a_1, \ldots, a_n)$: along lines, along parabolas, along spirals...

To show a limit does not exist, we can show two different ways of approach that lead to different limits. (This is like showing the right-hand and left-hand limits are unequal.) To show a limit equals L, you need to show the limit will be L on every possible approach. It is not enough to check some example approaches.

The theorems that should be true about the limits of sums, products, compositions (paying attention to continuity), etc., and the squeeze theorem, are true, and you can use them.

Examples:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}:$$

DNE (does not exist); approach on lines $x = 0$ or $y = 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + y^2}:$$

$= 1$, clearly.

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$:

DNE (Approach on lines $x = 0$ or $y = 0$ get 0, on $x = y$ get $\frac{1}{2}$.)

We can also use polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$,

$$\frac{xy}{x^2+y^2} = \frac{(r\cos\theta)(r\sin\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r^2(\cos\theta)(\sin\theta)}{r^2(\cos^2\theta+\sin^2\theta)} = (\cos\theta)(\sin\theta),$$

which approaches different limits as $(x, y) \to (0, 0)$ along different lines.

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$:

$$0 \leq \left|\frac{x^3}{x^2+y^2}\right| = |x| \left|\frac{x^2}{x^2+y^2}\right| \leq |x|.$$

Since

$$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} |x| = 0,$$

by the Squeeze Theorem we have

$$\lim_{(x,y)\to(0,0)} \left|\frac{x^3}{x^2+y^2}\right| = 0,$$

and therefore

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3}{x^2+y^2}\right) = 0.$$

We can also use polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$,

$$0 \leq \left|\frac{x^3}{x^2+y^2}\right| = \left|\frac{r^3\cos^3(\theta)}{r^2}\right| = r\left|\cos^3(\theta)\right| \leq r = \sqrt{x^2+y^2},$$

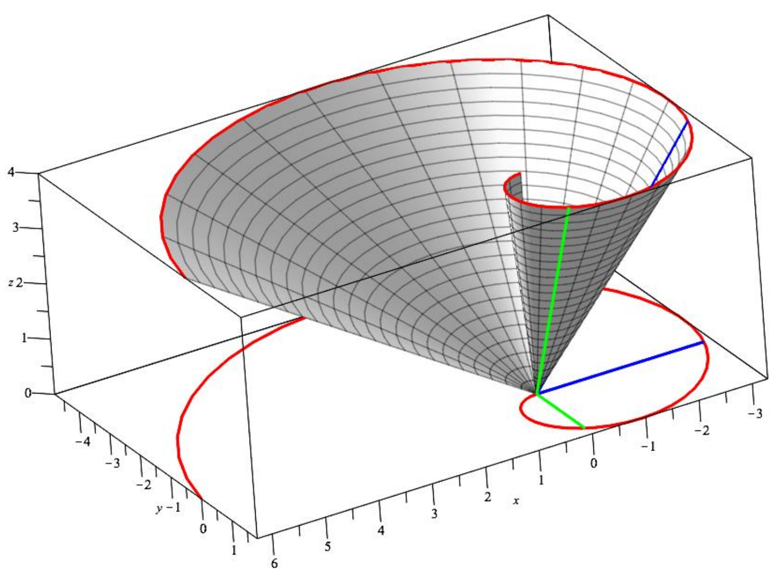
and apply the Squeeze Theorem as above.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}:$$

$$= \lim_{u\to 0} \frac{\sin(u)}{u} = 1 \text{ (use l'Hôpital's rule for the last step).}$$

$$\lim_{(x,y)\to(0,0)} \left(\frac{r}{\theta}\right) \text{ where } (r,\theta) \text{ are polar coordinates of } (x,y) \text{, chosen so } r \geq 0 \text{ and } 0 < \theta \leq 2\pi.$$

The limit does not exist. This is a weird example. Approaching the origin on any ray $\theta = a$, where a is constant, we have $\frac{r}{\theta} = \frac{r}{a} \to 0$, but approaching the origin along the spiral $r = \theta$, we have $\frac{r}{\theta} = \frac{r}{r} = 1 \to 1$.



This example demonstrates that checking limits as $(x_1, \ldots, x_n)$ approaches $(a_1, \ldots, a_n)$ along all possible straight lines is not sufficient.

Examples: (hints available)

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} \big((x^2 + y^2)\ln(x^2 + y^2)\big)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^6 + y^6}$$

An example from Monday: $\lim_{t\to 0}\left(t^2\ln(t^2)\right)$:

Method 1: Manipulate this into a form that allows us to use l'Hôpital's rule:

$$\lim_{t\to 0}\left(t^2\ln(t^2)\right) = \lim_{t\to 0}\frac{\ln(t^2)}{\frac{1}{t^2}} = \lim_{t\to 0}\frac{\frac{2t}{t^2}}{\frac{-2}{t^3}} = \lim_{t\to 0}\left(-t^2\right) = 0.$$

Method 2: Spoiler: This doesn't work, at least not immediately. (I made an error on the board in class, which is why it looked like it might work.) However, it is a natural technique to try to eliminate a problematic ln from something you are taking the limit of, so it's worth looking at.

Use the fact that e^x is a continuous function, and for any continuous function $g(x)$, we have (provided both limits are defined)

$$g(\lim_{t\to a} f(t)) = \lim_{t\to a} g(f(t)).$$

If you can evaluate the right-hand side, and you know the inverse of the function g, then you can evaluate the left-hand side. For example,

$$\left[\lim_{t\to a}\left(e^{f(t)}\right) = L\right] \implies \left[e^{\left(\lim_{t\to a} f(t)\right)} = L\right] \implies \left[\lim_{t\to a} f(t) = \ln(L)\right].$$

In our case, $f(t) = t^2\ln(t)$. Using exponent rules,

$$\lim_{t\to 0}\left(e^{\,t^2\ln(t^2)}\right) = \lim_{t\to 0}\left(\left(e^{\ln(t^2)}\right)^{t^2}\right) = \lim_{t\to 0}\left(\left(t^2\right)^{t^2}\right).$$

This limit is not obvious. If we are looking for the limit of a^b as $a \to 0$ and $b \to 0$, the fact that $a \to 0$ suggests that the limit is 0, but the fact that $b \to 0$ suggests that the limit is 1. In fact, now we have the tools to check that

$$\left(\lim_{(x,y)\to(0,0)} x^y\right) \text{ does not exist.}$$

If we are faced with finding $\lim_{t\to 0}\left(\left(t^2\right)^{t^2}\right)$, perhaps the best option is to use Method 2 with $\ln(x)$ in place of e^x, which leads to the problem we started with:

$$\lim_{t\to 0}\ln\left(\left(t^2\right)^{t^2}\right) = \lim_{t\to 0}\left(t^2\ln(t^2)\right) \underbrace{=0}_{\text{from Method 1 above}} = \ln(1).$$

And therefore:

$$\lim_{t\to 0}\left(\left(t^2\right)^{t^2}\right) = 1.$$