Math 11
Fall 2016
Section 1
Monday, September 26, 2016

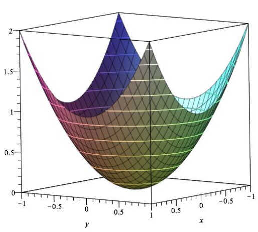
First, some important points from the last class:

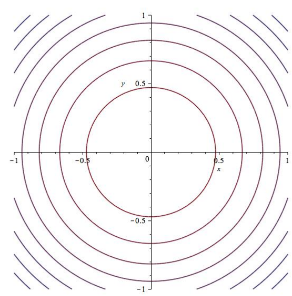
For $f : \mathbb{R}^2 \to \mathbb{R}$, the graph of f is a surface in $\mathbb{R}^3$, the set of all points $(x, y, f(x, y))$.

The level curves of f are curves $f(x, y) = k$ in $\mathbb{R}^2$. We can think of them as projections onto the xy-plane of horizontal slices of the graph of f.

We can draw a contour plot of f by drawing level curves $f(x, y) = k$ for equally spaced values of k. The contour plot is in $\mathbb{R}^2$ and is like a topographical map of the graph of f.

By looking at the contour plot we can see where the graph of f is steepest, what direction the surface slopes in, and where high and low points are.





For $f : \mathbb{R}^3 \to \mathbb{R}$, the graph of f is in $\mathbb{R}^4$, the set of all points $(x, y, z, f(x, y, z))$. We cannot draw it.

The level surfaces of f are surfaces $f(x, y, z) = k$ in $\mathbb{R}^3$. We can draw them.

If $f(x, y, z)$ is the temperature at (x, y, z), the level surfaces of f are isotherms. If f gives barometric pressure, the level surfaces are isobars.

For $f : \mathbb{R}^n \to \mathbb{R}$, the graph of f is in $\mathbb{R}^{n+1}$, the set of all points $(x, y, z, \ldots, f(x, y, z, \ldots))$. The level sets of f are in $\mathbb{R}^n$. They have equations $f(x, y, z, \ldots) = k$.

Level curves and level surfaces are two kinds of level sets.

Definition:

$$\lim_{(x,y,z)\to(x_0,y_0,z_0} f(x,y,z) = L$$

means for every $\varepsilon > 0$ [desired output accuracy] there is a $\delta > 0$ [required input accuracy] such that, for every (x,y,z),

$$\left[\underbrace{\overbrace{|(x,y,z)-(x_0,y_0,z_0)|}^{\text{distance between } (x,y,z) \text{ and } (x_0,y_0,z_0)} < \delta \ \ \& \ \ (x,y,z) \neq (x_0,y_0,z_0)}_{\text{within input accuracy}}\right] \implies \underbrace{|f(x,y,z)-L| < \varepsilon}_{\text{within output accuracy}}.$$

Definition: The function $f(x,y,z)$ is continuous at (x_0,y_0,z_0) if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0).$$

The definitions for $f : \mathbb{R}^2 \to \mathbb{R}$, and for $f : \mathbb{R}^n \to \mathbb{R}$, are similar.

If $F : \mathbb{R}^m \to \mathbb{R}^n$, we take limits coordinatewise. So if

$$F(x,y) = (F_1(x,y), F_2(x,y)),$$

then

$$\lim_{(x,y)\to(x_0,y_0)} F(x,y) = \left(\lim_{(x,y)\to(x_0,y_0)} F_1(x,y), \lim_{(x,y)\to(x_0,y_0)} F_2(x,y)\right).$$

To show a limit does not exist, we can show two different ways of approach that lead to different limits. (This is like showing the right-hand and left-hand limits are unequal.)

To show a limit exists (and equals L), it is not enough to check different approaches. Some tools you can use to show limits exist:

Breaking up an expression as a sum, product, composition...

The squeeze theorem.

Polar coordinates for limits as $(x,y) \to (0,0)$.

L'Hôpital's rule, but be warned that this is *only* if you have already reduced the problem to the limit of a function of one variable. This is an important warning. We do not have a two-dimensional version of L'Hôpital's rule.

Preview: (We will discuss differentiability later.)

Definition: If the graphs of $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}$ are tangent at the point (x_0, y_0, z_0), and

$$\mathcal{P}(x, y) = ax + by + d = \langle a, b \rangle \cdot \langle x, y \rangle + d$$

(in other words, the graph of $\mathcal{P}$ is a tangent plane to the graph of f), then we say f is differentiable at (x_0, y_0), and

$$\boxed{f'(x_0, y_0) = \langle a, b \rangle .}$$

Note: This is just like the case for $f : \mathbb{R} \to \mathbb{R}$. If the graph of the function

$$\ell(x) = ax + d$$

is the tangent line to the graph of f at the point (x_0, y_0), then the derivative of f at that point is the slope of that line:

$$\boxed{f'(x_0) = a.}$$

This is also just like the case for $\vec{f} : \mathbb{R} \to \mathbb{R}^n$: The tangent approximation to $\vec{f}$ at $t = t_0$ is

$$\vec{r}(t) = (t - t_0)\vec{v}_0 + \vec{r}_0 = t\vec{v}_0 + (\vec{r}_0 - t_0\vec{v})$$

where

$$\boxed{\vec{r}'(t_0) = \vec{v}_0.}$$

General idea: Suppose f is a function, and $\mathcal{T}$ is a function of the form

$$T(x) = Ax + D$$

where A and D are constants of the appropriate type (scalars or vectors), and multiplication can mean ordinary multiplication, scalar multiplication, or dot product, as appropriate. (The input x may also be a scalar or a vector.) If the graphs of f and $\mathcal{T}$ are tangent where $x = x_0$, then

$$\boxed{f'(x_0) = A.}$$

Warm-up problems:

(1.) Let $\mathcal{P}(x, y) = 4x + 2y - 5$. Find a vector parametric equation for ℓ the line of intersection of the graph of $\mathcal{P}$ and the plane $x = 2$. What is the slope (vertical rise over horizontal run, regarding the z-axis as vertical) of this line?

The graph of $\mathcal{P}$ has equation $z = 4x + 2y - 5$ Setting $x = 2$ and $y = t$ we have $z = 2t + 3$, so a vector parametric equation is

$$\vec{r} = \langle 2, t, 2t + 3\rangle = \langle 2, 0, 3\rangle + t\,\langle 0, 1, 2\rangle\,.$$

Since $\langle 1, 0, 2\rangle$ is parallel to ℓ, and has horizontal projection of length 1 and vertical projection of length 2, the slope of ℓ (regarding the z-axis as vertical) is 2.

(2.) Let $f(x, y) = x^2 + y^2$. Let $h(y) = f(2, y)$. Find $h'(1)$. What does this number say about the curve γ formed by intersecting the graph of f with the plane $x = 2$?

$h(y) = f(2, y) = 4 + y^2$ so $h'(y) = 2y$ and $h'(1) = 2$. Since γ is given by $x = 2$ and $z = f(2, y) = h(y)$, the derivative $h'(1)$ is the slope of γ (regarding the z-axis as vertical) when $y = 1$.

(3.) Show that $(2, 1, 5)$ lies on both ℓ and γ.

Putting $t = 1$ into the equation of ℓ gives $\langle 2, 1, 5\rangle$, so $(2, 1, 5)$ is on ℓ. From $f(2, 1) = 5$ we see $(2, 1, 5)$ is on the graph of f; since it is also in the plane $x = 2$, it is on γ.

(4.) What can we say about the geometric relation of ℓ and γ?

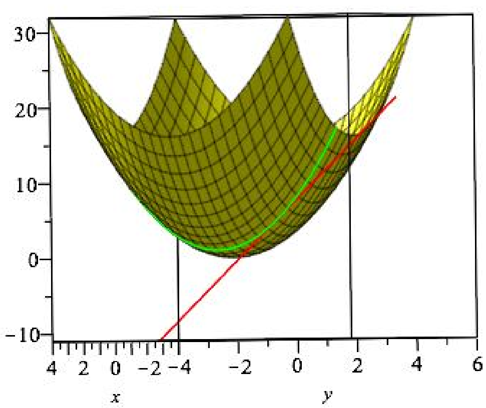
Both ℓ and γ lie in the plane $x = 2$, parallel to the yz plane, and contain the point $(2, 1, 5)$. At that point, ℓ has slope 2 (by (1)), and γ has slope 2 (by(2)). Since they lie in the same plane and have the same slope at the point $(2, 1, 5)$, they are tangent at that point.

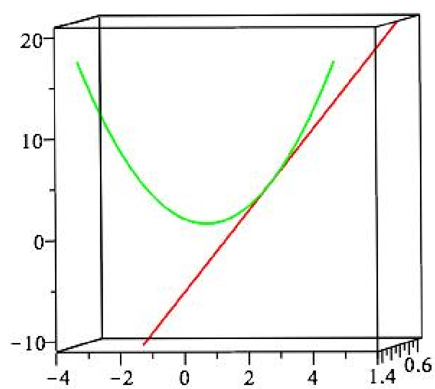
Today: Partial Derivatives

Example: Consider the surface S with equation $f(x,y) = x^2 + y^2$.
How can we describe the slope (treating the z-axis as vertical) of S?
As an example, consider the point $(2,1,5)$ on S.

If we slice S in the plane $x = 2$, we get a parabola, $z = 4 + y^2$, and we can compute the rate of change of z with respect to y when $y = 1$,

$$\frac{dz}{dy}\bigg|_{y=1} = \frac{d}{dy}(4 + y^2)\bigg|_{y=1} = (2y)\bigg|_{y=1} = 2$$





If we slice S in the plane $y = 1$, we get a parabola, $z = x^2 + 1$, and we can compute the rate of change of z with respect to x when $x = 2$,

$$\frac{dz}{dx}\bigg|_{x=2} = \frac{d}{dx}(x^2 + 1)\bigg|_{x=2} = (2x)\bigg|_{x=2} = 4$$

Geometrically, these are the slopes (vertical rise over horizontal run, treating the z-axis as vertical) of the tangent lines to S at $(2,1,5)$ in the planes $y = 1$ and $x = 2$.

These are the *partial derivatives* of $f(x,y)$ with respect to x and with respect to y.

Definition: The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is the derivative of the function of x we get by setting y to have constant value y_0:

$$\underbrace{\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = D_x f(x_0, y_0)}_{\text{notation}} = \frac{d}{dx}\left(f(x, y_0)\right)\Big|_{x=x_0}$$

$$= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Example: The partial derivatives of $f(x, y) = x^2 - y^2$, computed by treating the other variable as a constant, are

$$\frac{\partial f}{\partial x}(x, y) = 2x \qquad \frac{\partial f}{\partial y}(x, y) = -2y.$$

$$f_x(3, 1) = 6 \qquad f_y(3, 1) = -2.$$

If $z = f(x, y)$ we may also call the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Example: If $z = y \sin(xy)$ then

$$\frac{\partial z}{\partial x} = y^2 \cos(xy) \qquad \frac{\partial z}{\partial x} = \sin(xy) + xy \cos(xy)$$

Definition: The partial derivative of $f(x, y, z)$ with respect to x at the point (x_0, y_0, z_0) is the derivative of the function of x we get by setting y and z to have constant values y_0 and z_0:

$$\underbrace{\frac{\partial f}{\partial x}(x_0, y_0, z_0) = f_x(x_0, y_0, z_0) = D_x f(x_0, y_0, z_0)}_{\text{notation}} = \frac{d}{dx}\left(f(x, y_0, z_0)\right)\Big|_{x=x_0}.$$

Example: The partial derivatives of $f(x, y, z) = xyz$, computed by treating the other variables as a constant, are

$$\frac{\partial f}{\partial x}(x, y, z) = yz \qquad \frac{\partial f}{\partial y}(x, y, z) = xz \qquad \frac{\partial f}{\partial z}(x, y, z) = xy.$$

The partial derivative of f are themselves functions from $\mathbb{R}^2 \to \mathbb{R}$, and we can take their partial derivatives, called the second partial derivatives of f.

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Notice the order of x and y in the different notations.

Example: Find all the first and second partial derivatives.

$$f(x, y) = x^2 + 2xy - y^2$$

$$f_x(x, y) = 2x + 2y$$

$$f_{xx}(x, y) = 2 \qquad \boxed{f_{xy}(x, y) = 2}$$

$$f_y(x, y) = 2x - 2y$$

$$f_{yy}(y, x) = -2 \qquad \boxed{f_{yx}(x, y) = 2}$$

$$f(x, y) = e^x \sin(xy)$$

$$f_x(x, y) = e^x \sin(xy) + ye^x \cos(xy)$$

$$f_{xx}(x, y) = e^x \sin(xy) + ye^x \cos(x, y) + ye^x \cos(x, y) - y^2 e^x \sin(x, y)$$

$$\boxed{f_{xy}(x, y) = xe^x \cos(x, y) + e^x \cos(x, y) - xye^x \sin(x, y)}$$

$$f_y(x, y) = xe^x \cos(xy)$$

$$f_{yy}(x, y) = -x^2 e^x \sin(xy)$$

$$\boxed{f_{yx}(x, y) = e^x \cos(xy) + xe^x \cos(xy) - xye^x \sin(xy)}$$

Notice something?

Theorem (Clairaut's theorem): If suitable hypotheses hold (the first and second partial derivatives of f are continuous near the point in question), the corresponding mixed second partial derivatives of a function are always equal. That is,

$$f_{xy} = f_{yx} \qquad f_{xz} = f_{zx} \qquad f_{yz} = f_{zy}$$

Example: The motion of a vibrating string, anchored on the x-axis at points $x = a$ and $x = b$ and vibrating in the xy-plane, may be described by a function $f(x, t)$ giving the y-coordinate at time t of the point on the string with x-coordinate equal to x.

At a particular time t_0 and point on the string x_0, the physical significance of the first and second partial derivatives is:

$f_x(x, t)$ is he slope of the string at point x.

$f_t(x, t)$ is the velocity of point x on the string.

$f_{xx}(x, t)$ is the second derivative of the y-coordinate of the string (this determines the curvature).

$f_{xt}(x, t)$ is the rate at which the slope of the string is changing over time.

$f_{tx}(x, t)$ is the rate at which the instantaneous velocity, at a fixed time, changes with respect to distance along the string.

$f_{tt}(x, t)$ is the acceleration of point x on the string.

Why does it make sense that $f_{xy} = f_{yx}$?

Why does it make sense that in this physical situation f must satisfy the wave equation,

$$f_{tt} = c^2 f_{xx}$$

for some constant c?

Check that $f(x, t) = \sin(x)\cos(ct)$ satisfies this equation.

$$f_t(x, t) = -c\sin(x)\sin(ct) \qquad f_x(xt) = \cos(x)\cos(ct)$$
$$f_{tt}(x, t) = -c^2\sin(x)\cos(ct) \qquad f_{xx}(x, t) = -\sin(x)\cos(ct)$$
$$f_{tt} = -c^2\sin(x)\cos(ct) = c^2(-\sin(x)\cos(ct)) = c^2 f_{xx}.$$

We can use implicit differentiation to find partial derivatives.

Example: Find the slope (treating the z-axis as vertical) of the tangent line to the sphere $x^2 + y^2 + z^2 = 50$ at the point $(3, 4, 5)$ that lies in the plane $x = 3$.

Method 1: Write z as a function of x and y, and then find the partial derivative:

$$z = \pm\sqrt{50 - x^2 - y^2} = \sqrt{50 - x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(50 - x^2 - y^2)^{-\frac{1}{2}}(-2y) = -y(50 - x^2 - y^2)^{-\frac{1}{2}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(3,4)} = -4(50 - 9 - 16)^{-\frac{1}{2}} = \frac{-4}{5}.$$

Method 2: Implicitly differentiate the equation with respect to y. We are taking partial derivatives with respect to y, treating x as a constant, and z as a function of y.

$$x^2 + y^2 + z^2 = 50$$

$$0 + 2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-y}{z}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y,z)=(3,4,5)} = \frac{-4}{5}.$$

Example: Find a vector in the direction of this tangent line.

This line lies in the plane $x = 3$, goes through the point $(3, 4, 5)$, and has slope $\frac{-4}{5}$, so if y changes by 1, then z changes by $\frac{-4}{5}$ (and x does not change at all). This tells us the vector $\left\langle 0, 1, \frac{-4}{5} \right\rangle$ gives the direction of the line, so an equation is

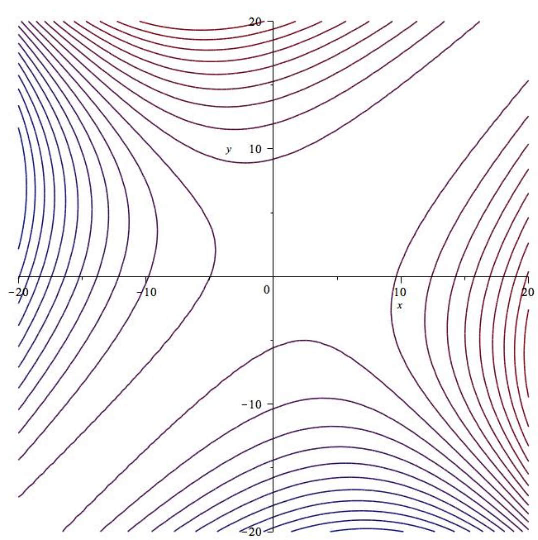
$$\langle x, y, z \rangle = \langle 3, 4, 5 \rangle + t\left\langle 0, 1, \frac{-4}{5} \right\rangle.$$

Example: Suppose that $f(x, y)$ denotes the average temperature at points on the earth whose latitude is x and altitude is y, if we identify latitudes north of the equator as positive and south of the equator as negative.

For what values of (x, y) do you expect f_x to be positive? For what values of (x, y) do you expect f_x to be negative? Why?

For what values of (x, y) do you expect f_y to be positive? For what values of (x, y) do you expect f_y to be negative? Why?

Example: Below is a contour plot for a function $f(x, y)$. The value of f increases in the positive direction along both the x-axis and the y-axis. Compare the partial derivatives of f at the points $(-10, 15)$, $(0, 15)$ and $(15, -15)$. (At which of these points are f_x and f_y largest or smallest; positive or negative?)



Example: The surface S has equation $z = x^2y - y^2x$. Find the line that lies in the plane $y = 2$ and is tangent to S at the point $(1, 2, -2)$.

Example: Show that the function

$$f(x, y) = 5e^{3x+1}\sin(3y - 4)$$

satisfies Laplace's equation,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Example: We saw that the direction of the line in the plane $x = 3$ tangent to the sphere $x^2 + y^2 + z^2 = 50$ at the point $(3, 4, 5)$ is given by the vector $\left\langle 0, 1, \frac{-4}{5} \right\rangle$.

Find a vector giving the direction of the line in the plane $y = 4$ tangent to the sphere $x^2 + y^2 + z^2 = 50$ at the point $(3, 4, 5)$.

Use this information to find an equation for the plane that is tangent to the sphere $x^2 + y^2 + z^2 = 50$ at the point $(3, 4, 5)$.