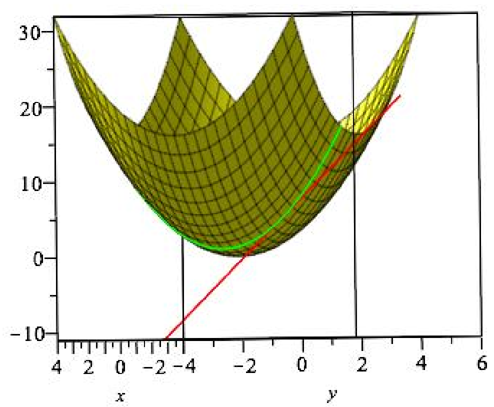
Math 11
Fall 2016
Section 1
Wednesday, September 28, 2016

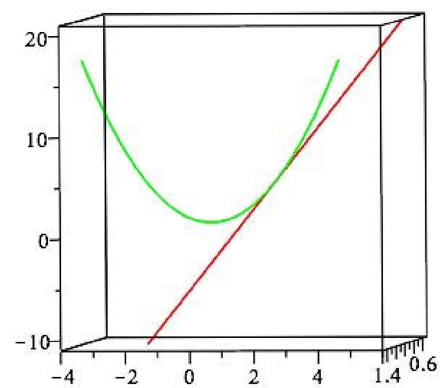
First, some important points from the last class:

Definition: The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is the derivative of the function of x we get by setting y to have constant value y_0:

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = D_x f(x_0, y_0) = \frac{d}{dx}\left(f(x, y_0)\right)\Big|_{x=x_0}.$$

Geometrically, this is the slope (vertical rise over horizontal run, treating the z-axis as vertical) of the tangent line to the graph of f at $(x_0, y_0, f(x_0, y_0)))$ in the plane $x = x_0$.





The second partial derivatives of f include

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

Theorem (Clairaut's theorem): If suitable hypotheses hold, the corresponding mixed second partial derivatives of a function are always equal. That is,

$$f_{xy} = f_{yx} \qquad f_{xz} = f_{zx} \qquad f_{yz} = f_{zy}$$

Example: Find an equation for the tangent plane to the graph of the function

$$f(x,y) = x^2y^2$$

at the point $(1,3,9)$.

The partial derivatives of f at that point are

$$\frac{\partial f}{\partial x}(1,3) = (2xy^2)\Big|_{(x,y)=(1,3)} = 18$$

$$\frac{\partial f}{\partial y}(1,3) = (2x^2y)\Big|_{(x,y)=(1,3)} = 6$$

Vectors in the direction of the lines tangent to the graph of f at that point in vertical planes:

$$x = 1: \qquad \left\langle 0, 1, \frac{\partial f}{\partial y}(1,3) \right\rangle = \langle 0,1,6\rangle$$

$$y = 3: \qquad \left\langle 1, 0, \frac{\partial f}{\partial x}(1,3) \right\rangle = \langle 1,0,18\rangle$$

Vector normal to both tangent lines:

$$\langle 0,1,6\rangle \times \langle 1,0,18\rangle = \langle 18,6,-1\rangle$$

Equation of plane containing both tangent lines (containing point $(1,3,9)$ and normal to the vector $\langle 18,6,-1\rangle$):

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\langle 18,6,-1\rangle \cdot \langle x-1, y-3, z-9\rangle = 0$$

$$18(x-1) + 6(y-3) - (z-9) = 0$$

$$z = 18(x-1) + 6(y-3) + 9$$

$$z = \left(\frac{\partial f}{\partial x}(1,3)\right) \underbrace{(x-1)}_{\Delta x} + \left(\frac{\partial f}{\partial xy}(1,3)\right) \underbrace{(y-3)}_{\Delta y} + f(1,3)$$

Theorem: If the graph of f has a tangent plane at the point $(x_0, y_0, f(x_0,y_0))$, its equation is

$$z = \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x-x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y-y_0) + f(x_0,y_0).$$

Theorem: If the graph of f has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$, it is the graph of the function

$$L(x, y) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0).$$

We can approximate $f(x, y)$ near (x_0, y_0) by

$$f(x, y) \approx L(x, y)$$

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(\Delta x) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(\Delta y) + f(x_0, y_0).$$

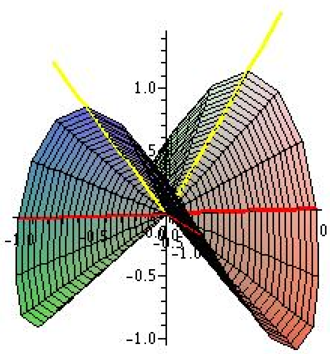
This is called the *linear approximation* or *tangent approximation.*

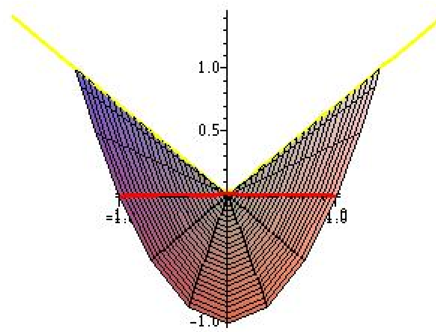
Definition: The function

$$L(x, y) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0)$$

is called the *linearization* of f at (x_0, y_0).

Warning: The fact that f has partial derivatives at a point is *not enough* to guarantee that its graph has a tangent plane there. Here are two pictures of the graph of the function





$f(x, y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0); \\ 0 & (x, y) = (0, 0). \end{cases}$ The red lines are the intersections of the graph of f with the planes $x = 0$ and $y = 0$. Both are horizontal, so $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$. The yellow V is the intersection of the graph of f with the plane $x = y$. It is pointed at the origin, and does not have a tangent line there, so the graph of f has no tangent plane at $(0, 0)$.

We do, however, have this useful theorem:

Theorem: If the partial derivatives of $f(x, y)$ are defined near (x_0, y_0) and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Example: Show that

$$f(x, y, z) = xyz$$

is differentiable at the point $(1, 2, 1)$, and then use the linear approximation to f to approximate the product of the three numbers 1.01, 1.98, and .99.

The partial derivatives of f are

$$\frac{\partial f}{\partial x}(x, y, z) = yz \qquad \frac{\partial f}{\partial y}(x, y, z) = xz \qquad \frac{\partial f}{\partial z}(x, y, z) = xy.$$

They are defined and continuous everywhere, so by the theorem, f is differentiable everywhere.

For small values of Δx, Δy, and Δz, we can say

$$f(1 + \Delta x, 2 + \Delta y, 1 + \Delta z) \approx$$
$$\left(\frac{\partial f}{\partial x}(1, 2, 1)\right) \Delta x + \left(\frac{\partial f}{\partial y}(1, 2, 1)\right) \Delta y + \left(\frac{\partial f}{\partial z}(1, 2, 1)\right) \Delta z + f(1, 2, 1) =$$
$$2\Delta x + \Delta y + 2\Delta z + 2.$$

At the point $(1.01, 1.98, .99)$, we have $\Delta x = .01$, $\Delta y = -.02$ and $\Delta z = -.01$, so

$$(1.01)(1.98)(.99) = f(1.01, 1.98, .99) \approx 2(.01) + (-.02) + 2(-.01) + 2 = 1.98$$

(The actual product, per calculator, is 1.979802. Our error is .000198, which is about .01%. This seems pretty good, since Δx, Δy, and Δz were about 1% of our original numbers.)

We can say:

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \approx \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(\Delta x) + \left(\frac{\partial f}{\partial xy}(x_0, y_0)\right)(\Delta y);$$

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$$

Definition: The differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \quad \text{or} \quad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Example: Find an equation for the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 169$$

at the point $(3, 4, 12)$.

We can consider z to be a function of x and y on the top half of the sphere, so $z = f(x, y)$. The graph of the linearization of f at $(3, 4)$ will be tangent to the graph of f. We can find $\frac{\partial z}{\partial x}$ by implicit differentiation, treating y as a constant, z as a function of x, and x as the independent variable:

$$x^2 + y^2 + z^2 = 169$$

$$2x + 0 + 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

In the same way, we get

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

and at $(x, y) = (3, 4)$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = -\frac{3}{12} \qquad \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = -\frac{4}{12}$$

Our linearization is

$$L(x, y) = \frac{\partial f}{\partial x}(3, 4)(x - 3) + \frac{\partial f}{\partial y}(3, 4)(y - 4) + f(3, 4) =$$

$$\left(\frac{-3}{12}\right)(x - 3) + \left(\frac{-4}{12}\right)(y - 4) + 12 = -\frac{x}{4} - \frac{y}{3} + \frac{169}{12}$$

so we can write our tangent plane as

$$z = -\frac{x}{4} - \frac{y}{3} + \frac{169}{12}$$

$$3x + 4y + 12z = 169.$$

Recall from last time we made the following definition:

Definition: The function $f(x, y)$ is differentiable at (x_0, y_0) if there is a function

$$L(x, y) = ax + by + c$$

(where a, b, and c are constants) whose graph is tangent to the graph of f at the point $(x_0, y_0, f(x_0, y_0)) = (x_0, y_0, z_0)$.

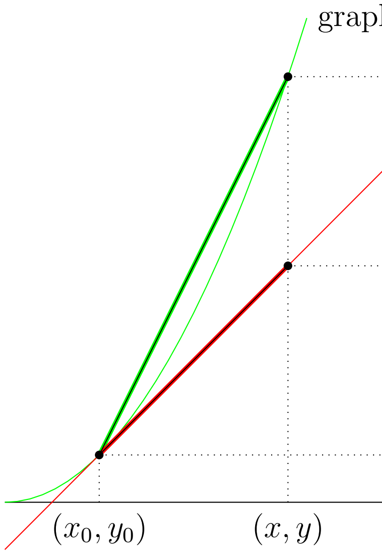
We now know what the function L must be:

$$L(x, y) = \left(\frac{\partial f}{\partial x}\right)(x_0, y_0)(x - x_0) + \left(\frac{\partial f}{\partial y}\right)(x_0, y_0)(y - y_0) + f(x_0, y_0).$$

It remains to say what it means for the graphs of L and f to be tangent at (x_0, y_0, z_0).

The first condition is obvious; the point (x_0, y_0, z_0) must be on both graphs. That is, we must have $z_0 = f(x_0, y_0) = L(x_0, y_0)$.

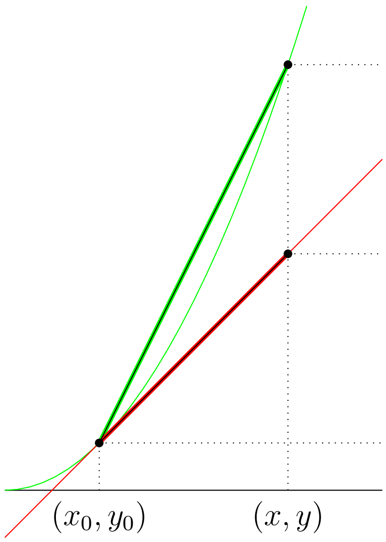
The second condition is more complex. Intuitively, we want the two graphs to have the same slopes at that point. But we saw that a graph can have different slopes in different directions. So, again intuitively, we want the graphs to have the same slope in every direction.



(Vertical slice of graphs of f (green) and L (red).)

We want the slope of a secant line to the graph of f to be close to the slope of the corresponding line segment in the graph of L, as long as (x, y) is close to (x_0, y_0). We would like to say that in the limit as $(x, y) \to (x_0, y_0)$ the slopes are the same — except that in almost all cases neither slope approaches a limit. (For example, as $(x, y) \to (x_0, y_0)$ from the positive x-direction, the slope of the secant line approaches $f_x(x_0, y_0)$, and as $(x, y) \to (x_0, y_0)$ from the positive y-direction, the slope of the secant line approaches $f_y(x_0, y_0)$.)

However, the following small change works: We require that the limit of the *difference* of the slopes as $(x, y) \to (x_0, y_0)$ is zero.



From the picture, the horizontal "run" of each line segment is the distance between (x_0, y_0) and (x, y), or $\sqrt{(x-x_0)^2 + (y-y_0)^2}$. The vertical "rise" of the secant line to the graph of f is $f(x,y) - f(x_0,y_0)$, and that of the line segment in the graph of L is $L(x,y) - L(x_0,y_0)$. Their respective slopes are

$$slope_{f-secant} = \frac{f(x,y) - f(x_0,y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \qquad slope_{L-line} = \frac{L(x,y) - L(x_0,y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}},$$

and using the fact that $f(x_0,y_0) = L(x_0,y_0)$, the difference of those slopes is

$$\frac{(f(x,y) - f(x_0,y_0)) - (L(x,y) - L(x_0,y_0))}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = \frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}.$$

The graphs are tangent if that difference approaches 0 as $(x, y) \to (x_0, y_0)$. Putting this together:

Definition: The function $f(x, y)$ is differentiable at (x_0, y_0) if there is a function

$$L(x,y) = ax + by + c$$

(where a, b, and c are constants) such that $L(x_0,y_0) = f(x_0,y_0)$ and

$$\lim_{(x,y)\to(x_0,y_0)} \left(\frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \right) = 0.$$

Warning: This is not the same as the definition in the textbook. Both definitions say that the difference $f(x, y) = L(x, y)$ not only approaches zero as $(x, y) \to (x_0, y_0)$, it approaches zero very quickly. You can use either one. (See the mathematical challenge problem at the end.)

Example: Let $f(x, y) = xy$.

Show that f is differentiable everywhere.

Find an equation for the plane that is tangent to the graph of the function $f(x, y) = xy$ at the point $(1, 1, 1)$.

Use the linearization of f at $(x, y) = (1, 1)$ to approximate the value of the product $(1.02)(.97)$.

Example: Use implicit differentiation to find the partial derivatives of z with respect to x and y on the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 3$$

at the point $(3, 4, 5)$. Then find an equation for the tangent plane to the ellipsoid at that point.

Use the linear approximation to approximate the z-coordinate of the point on the ellipsoid whose x- and y-coordinates are 3.02 and 4.01.

Example: Show that any function of the form

$$f(x, y) = ae^{bx} \sin(by),$$

where a and b are constants, satisfies Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Example: Check directly that Clairaut's Theorem holds of any function of the form

$$f(x, y) = g(x)h(y),$$

where g and h are differentiable functions.

Example: Let $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0); \\ 0 & (x,y) = (0,0). \end{cases}$. This is the example shown above of a function that has partial derivatives but is not differentiable.

Show that f is continuous at $(0,0)$.

Show that $\frac{\partial f}{\partial x}(0,0) = 0$. Because of the piecewise definition of f, you should do this using the limit definition of partial derivative,

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

(By symmetry, we also have $\frac{\partial f}{\partial y}(0,0) = 0$.)

Compute $\frac{\partial f}{\partial x}(x,y)$ for $(x,y) \neq (0,0)$, and show that $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$.

Mathematical Challenge: Show that if f is differentiable at (x_0, y_0) according to the textbook definition, then it is differentiable at (x_0, y_0) according to our definition.