Math 11
Fall 2016
Section 1
Friday, September 30, 2016

First, some important points from the last class:

**Definition:** The function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0, y_0)$ if there is a function

$$L(x, y) = ax + by + c$$

(where $a$, $b$, and $c$ are constants) such that the graphs of $f$ and $L$ are tangent at the point $(x_0, y_0, f(x_0, y_0))$. This means that

$$L(x_0, y_0) = f(x_0, y_0)$$

and

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

**Theorem:** If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0, y_0)$, then the tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ is the graph of the function

$$L(x, y) = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial xy}(x_0, y_0)\right)(y - y_0) + f(x_0, y_0).$$

**Note:** When $f$ is differentiable at $(x_0, y_0)$, we can approximate $f(x, y)$ near $(x_0, y_0)$ by

$$f(x, y) \approx L(x, y).$$

This is called the *linear approximation* or *tangent approximation* to $f$ near $(x_0, y_0)$. The function $L(x, y)$ is called the *linearization* of $f$ at $(x_0, y_0)$.

**Definition:** If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, the *differential* of $f$ is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

**Theorem:** If the partial derivatives of $f(x, y)$ are defined on a neighborhood of $(x_0, y_0)$ and continuous at $(x_0, y_0)$, then $f$ is differentiable at $(x_0, y_0)$.

**Note:** These definitions and theorems also hold for $f : \mathbb{R}^3 \to \mathbb{R}$, and in general for $f : \mathbb{R}^n \to \mathbb{R}$.

**Warm-up Problems:**

A surface $S$ has the equation $z = f(x, y)$. At $(x, y) = (1, 2)$ we have

$$z = 45 \qquad \frac{\partial z}{\partial x} = 2 \qquad \frac{\partial z}{\partial y} = 1$$

A bug is crawling on the surface $S$, and a light shining directly down through $S$ (which is transparent) casts the bug's shadow on the $xy$-plane; the position of the shadow is $\vec{r}(t)$. At time $t_0$, the bug's shadow has position $\vec{r}(t_0) = \langle 1, 2\rangle$ and velocity $\vec{r}\,'(t_0) = \langle 3, -1\rangle$.

1. Find an equation for the tangent plane to $S$ at the point $(1, 2, 45)$.

$$z = 45 + (2)(x-1) + (1)(y-2).$$

2. Use the tangent approximation

$$\vec{r}(t) \approx \vec{r}(t_0) + (t - t_0)\vec{r}\,'(t_0)$$

   to approximate the shadow's position at time $t_0 + \Delta t$, where $\Delta t$ is a very small change in $t$.

$$\vec{r}(t_0 + \Delta t) \approx \langle 1, 2\rangle + \Delta t\,\langle 3, -1\rangle = \langle 1 + 3\Delta t, 2 - \Delta t\rangle = \left\langle 1 + \underbrace{\overbrace{(3)}^{\frac{dx}{dt}}\,\Delta t}_{\Delta x},\ 2 + \underbrace{\overbrace{(-1)}^{\frac{dy}{dt}}\,\Delta t}_{\Delta y} \right\rangle.$$

3. Use the equation of the tangent plane to $S$ to approximate the bug's new $z$-coordinate.

$$z \approx 45 + 2(x-1) + 1(y-2) \approx 45 + 2((1 + 3\Delta t) - 1) + 1((2 - \Delta t) - 2) =$$

$$45 + (2)(3)\Delta t + (1)(-1)\Delta t = 45 + \underbrace{\underbrace{(2)}_{\frac{\partial z}{\partial x}}\ \underbrace{\overbrace{(3)}^{\frac{dx}{dt}}\,\Delta t}_{\Delta x} + \underbrace{(1)}_{\frac{\partial z}{\partial y}}\ \underbrace{\overbrace{(-1)}^{\frac{dy}{dt}}\,\Delta t}_{\Delta y}}_{\Delta z}$$

We can rewrite this:

$$f(\vec{r}(t_0 + \Delta t)) \approx 45 + \Big( \underbrace{\langle 2, 1\rangle}_{\left\langle \frac{\partial f}{\partial x}(1,2),\ \frac{\partial f}{\partial y}(1,2) \right\rangle} \cdot \underbrace{\langle 3, -1\rangle}_{\vec{r}\,'(t_0)} \Big)\Delta t.$$

$$f(\vec{r}(t_0 + \Delta t)) \approx f(\vec{r}(t_0)) + \left( \left\langle \frac{\partial z}{\partial x}(\vec{r}(t_0)),\ \frac{\partial z}{\partial y}(\vec{r}(t_0)) \right\rangle \cdot \vec{r}\,'(t_0) \right) \Delta t.$$

$$\boxed{\frac{f(\vec{r}(t_0 + \Delta t)) - f(\vec{r}(t_0))}{\Delta t} \approx \left( \left\langle \frac{\partial z}{\partial x}(\vec{r}(t_0)),\ \frac{\partial z}{\partial y}(\vec{r}(t_0)) \right\rangle \cdot \vec{r}\,'(t_0) \right)}.$$

**Definition:** If $f : \mathbb{R}^n \to \mathbb{R}$, the *gradient* of $f$ is the vector whose components are its partial derivatives:

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}(x,y,z),\ \frac{\partial f}{\partial y}(x,y,z),\ \frac{\partial f}{\partial z}(x,y,z) \right\rangle.$$

If $f$ is differentiable, we may also calll $\nabla f$ the total derivative of $f$.

**Theorem** (the chain rule): If $\vec{r}(t)$ is differentiable at $t_0$, and $f(x,y,z)$ is differentiable at $\vec{r}(t_0)$, then

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Rephrasing this, if $w$ is a function of $x$, $y$, $z$, and $x$, $y$, $z$ are all functions of $t$, then

$$\frac{dw}{dt} = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\Delta w \approx \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z \approx \frac{\partial w}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial w}{\partial y}\frac{dy}{dt}\Delta t + \frac{\partial w}{\partial z}\frac{dz}{dt}\Delta t$$

**Example** If $w = x^2y^2$, $x = \sin(t)$, and $y = \cos(t)$, find $\dfrac{dw}{dt}$ at $t = \dfrac{\pi}{3}$.

$$t = \frac{\pi}{3} \qquad x = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad y = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\frac{\partial w}{\partial x} = 2xy^2 = \frac{\sqrt{3}}{4} \qquad \frac{\partial w}{\partial y} = 2x^2y = \frac{3}{4} \qquad \frac{dx}{dt} = \cos(t) = \frac{1}{2} \qquad \frac{dy}{dt} = -\sin(t) = -\frac{\sqrt{3}}{2}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{4}$$

The chain rule in different settings:

$$t \to x \to w$$
$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

$$t \to (x,y,z) \to w$$
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$(s,t) \to (x,y,z) \to w$$
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

**Example:** Suppose $g(x, y) = f(x^2 - y^2, y^2 - x^2)$. Show that $g$ satisfies the differential equation

$$y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = 0.$$

Introduce new variables $s = x^2 - y^2$ and $t = y^2 - x^2$, and write $w = f(s, t)$. We want to show that

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0.$$

Using the Chain Rule,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial w}{\partial s}(2x) + \frac{\partial w}{\partial t}(-2x).$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial w}{\partial s}(-2y) + \frac{\partial w}{\partial t}(2y).$$

Now

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = y\left(\frac{\partial w}{\partial s}(2x) + \frac{\partial w}{\partial t}(-2x)\right) + x\left(\frac{\partial w}{\partial s}(-2y) + \frac{\partial w}{\partial t}(2y)\right) = 0.$$

Ways to visualize the Chain Rule:

Suppose $\vec{r} : \mathbb{R} \to \mathbb{R}^3$, and $f : \mathbb{R}^3 \to \mathbb{R}$. You can visualize $\vec{r}(t)$ as the position at time $t$ of a moving object, and $f(x, y, z)$ as the temperature at point $(x, y, z)$. Then the composition $(f \circ \vec{r})(t)$ represents the temperature of the moving object at time $t$ (assuming the object acquires the temperature of its surroundings), and its derivative $(f \circ \vec{r})'(t)$ represents the rate of change of the object's temperature with respect to time.

If we write $\vec{r} = \langle x, y, z \rangle$ and $w = f(x, y, z)$, then $w$ denotes the object's temperature, and the Chain Rule can be written as

$$\frac{dw}{dt} = (f \circ \vec{r})'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle =$$

$$\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

which gives us

$$dw = \frac{\partial w}{\partial x}\frac{dx}{dt}dt + \frac{\partial w}{\partial y}\frac{dy}{dt}dt + \frac{\partial w}{\partial z}\frac{dz}{dt}dt$$

$$\Delta w \approx \frac{\partial w}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial w}{\partial y}\frac{dy}{dt}\Delta t + \frac{\partial w}{\partial z}\frac{dz}{dt}\Delta t$$

We can think that a small change $\Delta t$ in time produces a small change $\Delta x \approx \frac{dx}{dt}\Delta t$ in $x$, which in turn produces a small change of approximately $\frac{\partial w}{\partial x}\Delta x \approx \frac{\partial w}{\partial x}\frac{dx}{dt}\Delta t$ in $w$. The change $\Delta t$ in $t$ also produces changes $\Delta y$ in $y$ and $\Delta z$ in $z$, and those changes also produce changes in $w$. The net change $\Delta w$ is the sum of the three individual changes produced by the changes in $x$, $y$, and $z$.

If $\vec{r} : \mathbb{R} \to \mathbb{R}^2$, and $f : \mathbb{R}^2 \to \mathbb{R}$, we can think of $\vec{r}(t)$ as the projection on the $xy$-plane of the position of an object (crawling bug) moving on the graph $z = f(x, y)$. Then the composition $(f \circ \vec{r})(t)$ represents the height ($z$-coordinate) of the moving object at time $t$, and its derivative $(f \circ \vec{r})'(t)$ represents the rate of change of the object's height with respect to time; that is, how fast its height is changing.

We can again think that changing $t$ produces changes in both $x$ and $y$, each of which contribute to change in $z$, and the net change in $z$ is the sum of those two changes.

**Example:** We can identify points on the cone $x^2+y^2=z^2$, $z \geq 0$, using two coordinates, $r$ and $\theta$, by setting

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = r \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq r.$$

Define $w$ on the cone by

$$w = xy - xz^2.$$

Note that we can think of $w$ as function of $r$ and $\theta$: $w = xy - xz^2$, where $(x, y, z)$ is the point on the cone for which $(x, y) = (r\cos(\theta), r\sin(\theta))$.

Find $\dfrac{\partial w}{\partial r}$ at the point $(-2, 0, 2)$.

At the point $(x, y) = (-2, 0)$ we have

$$r = 2 \qquad \theta = \pi \qquad x = -2 \qquad y = 0 \qquad z = 2$$

$$\frac{\partial w}{\partial x} = y - z^2 = -4 \qquad \frac{\partial w}{\partial y} = x = -2 \qquad \frac{\partial w}{\partial z} = -2xz = 8$$

$$\frac{\partial x}{\partial r} = \cos(\theta) = -1 \qquad \frac{\partial y}{\partial r} = \sin(\theta) = 0 \qquad \frac{\partial z}{\partial r} = 1$$

We treat $\theta$ as a constant and differentiate with respect to $r$, using the chain rule:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (-4)(-1) + (-2)(0) + (8)(1) = 12$$

At a general point, we have

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y - z^2)(\cos(\theta)) + (x)(\sin(\theta)) + (-2xz)(1) =$$

$$(r\sin(\theta) - r^2)(\cos(\theta)) + (r\cos(\theta))(\sin(\theta)) + (-2r^2\cos(\theta))(1) =$$

$$2r\sin(\theta)\cos(\theta) - 3r^2\cos(\theta).$$

What does this mean? We define $w$ as a function of $(r, \theta)$ by looking at the point on the cone $(x, y, z) = (r\cos(\theta), r\sin(\theta), r)$, then computing $w = xy - xz^2$. We want to know, when $(x, y) = (-2, 0)$, the rate of change of $w$ with respect to $r$.

For example, suppose $w$ denotes the temperature at a given point on the cone. Consider the ubiquitous bug crawling on the cone, with its shadow moving in the $xy$-plane. The bug's temperature is $w$. When the bug's shadow is where $(r, \theta) = (2, -\pi)$, and the bug moves so its shadow's new location is where $(r, \theta) = (2 + \Delta r, -\pi)$ (that is, $\theta$ remains constant and $r$ changes by $\Delta r$), the bug's temperature will have changed by

$$\Delta w \approx \frac{\partial w}{\partial r}\Delta r$$

**Example:** We can identify points on the cone $x^2+y^2=z^2$, $z \geq 0$, using two coordinates, $r$ and $\theta$, by setting

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = r \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq r.$$

Define $w$ on the cone by

$$w = xy - xz^2.$$

Note that we can think of $w$ as function of $r$ and $\theta$: $w = xy - xz^2$, where $(x, y, z)$ is the point on the cone for which $(x, y) = (r\cos\theta, r\sin\theta)$.

Find $\dfrac{\partial w}{\partial \theta}$ at the point $(-2, 0, 2)$.

**Example:** A surface $S$ has the equation $z = f(x, y)$. At $(x, y) = (1, 2)$ we have

$$z = 45 \qquad \frac{\partial z}{\partial x} = 2 \qquad \frac{\partial z}{\partial y} = 1$$

A bug is crawling on the surface $S$, and a light shining directly down through $S$ (which is transparent) casts the bug's shadow on the $xy$-plane; the position of the shadow is $\vec{r}(t)$. At time $t_0$, the bug's shadow has position $\vec{r}(t_0) = \langle 1, 2 \rangle$ and velocity $\vec{r}\,'(t_0) = \langle 3, -1 \rangle$.

Find the rate of change of the bug's altitude with respect to time at the time $t_0$.

**Example:**

Suppose that $S$ is a level surface $f(x, y, z) = k$ of a differentiable function $f$ and $\vec{r}(t)$ is a regular parametrization of a path $\gamma$ lying in $S$. Since the value of $f$ equals $k$ for all points on $S$, and all points $\vec{r}(t)$ are on $S$, we have

$$f(\vec{r}(t)) = k.$$

Start with this equation and differentiate both sides (using the chain rule for the left hand side) to show that

$$\nabla f(\vec{r}(t)) \perp \vec{r}\,'(t).$$

Since this is true for any path $\gamma$ in $S$, we can conclude that $\nabla f(x, y, z)$ is normal to $S$ at the point $(x, y, z)$. Explain why.

That is, the gradient of $f$ at a point is normal to the level surface (or level curve) of $f$ containing that point.

**Example:** If $f(x, y) = 4x^2 - y^2$, then the hyperbola $4x^2 - y^2 = 3$ is a level curve of $f$, so it should be perpendicular to the gradient of $f$ at every point. Verify that the hyperbola is perpendicular to the gradient of $f$ at the point $(1, 1)$ in the following way:

Use implicit differentiation to compute $\frac{dy}{dx}$ for the portion of the hyperbola containing $(1, 1)$, use the value of $\frac{dy}{dx}$ to find a vector $\vec{T}$ tangent to the hyperbola at $(1, 1)$, and then verify that $\vec{T}$ is perpendicular to $\nabla f(1, 1)$.

Recall implictly defined functions: An equation $f(x, y, z) = 0$ defining a surface $S$ can be thought of as implicitly defining $z$ as a function of $x$ and $y$ near a point on $S$. If we want to find $\frac{\partial z}{\partial x}$ at that point, we can treat $y$ as a constant and $z$ as a function of $x$, and differentiate the equation wtih respect to $x$:

$$\frac{\partial}{\partial x}(f(x, y, z)) = 0$$

$$\frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial f}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial f}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{\text{unknown}} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

This is the implicit function theorem.

**Example:** Earlier, we looked at the surface

$$ax^2 + by^2 + cz^2 = d$$

and used implicit differentiation:

$$2ax + 2cz\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{ax}{cz}.$$

Now we can use the implicit function theorem:

$$f(x, y, z) = ax^2 + by^2 + cz^2 - d \qquad f(x, y, z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{2ax}{2cz} = -\frac{ax}{cz}.$$

You do not have to know the implicit function theorem, but you may use it if you wish.

Proof of the Chain Rule:

If $f$ is differentiable at $(x_0, y_0)$, we can write

$$f(x,y) = \underbrace{a(x-x_0) + b(y-y_0) + f(x_0,y_0)}_{\mathcal{P}(x,y) \text{ graph is tangent plane}} + \underbrace{E(x,y)}_{f(x,y)-\mathcal{P}(x,y)}$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{|\langle x-x_0, y-y_0\rangle|} = 0$$

$$a = \frac{\partial f}{\partial x}(x_0,y_0) \qquad b = \frac{\partial f}{\partial y}(x_0,y_0) \qquad \nabla f(x_0,y_0) = \langle a, b\rangle$$

If $\vec{r}(t_0) = (x_0, y_0)$ and $\vec{r}$ is differentiable at $t_0$, we can write $\vec{r}(t) = \langle x(t), y(t)\rangle$ and compute

$$\frac{d}{dt}\left(f(\vec{r}(t_0))\right) = \lim_{t\to t_0} \frac{f(\vec{r}(t)) - f(\vec{r}(t_0))}{t-t_0} = \lim_{t\to t_0} \frac{f(x(t),y(t)) - f(x_0,y_0)}{t-t_0} =$$

$$\lim_{t\to t_0} \frac{a(x(t)-x_0) + b(y(t)-y_0) + f(x_0,y_0) + E(x(t),y(t)) - f(x_0,y_0)}{t-t_0} =$$

$$\lim_{t\to t_0} \frac{a(x(t)-x(t_0)) + b(y(t)-y(t_0)) + E(x(t),y(t))}{t-t_0}$$

$$= a\lim_{t\to t_0} \frac{(x(t)-x(t_0))}{t-t_0} + b\lim_{t\to t_0} \frac{(y(t)-y(t_0))}{t-t_0} + \lim_{t\to t_0} \frac{E(x(t),y(t))}{t-t_0} =$$

$$ax'(t_0) + by'(t_0) + \lim_{t\to t_0} \frac{E(x(t),y(t))}{t-t_0} =$$

$$\langle a,b\rangle \cdot \langle x'(t_0), y'(t_0)\rangle) + \lim_{t\to t_0} \frac{E(x(t),y(t))}{t-t_0} = \boxed{\nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)} + \lim_{t\to t_0} \frac{E(x(t),y(t))}{t-t_0}$$

Now, assuming for simplicity that $\vec{r}\,'(t_0) \neq \vec{0}$, so that for $t$ near $t_0$ we have $\vec{r}(t) \neq \vec{r}(t_0)$ and we can safely divide by $|\vec{r}(t) - \vec{r}(t_0)|$ (this assumption can be eliminated by a small trick),

$$\lim_{t\to t_0} \left|\frac{E(x(t),y(t))}{t-t_0}\right| = \lim_{t\to t_0} \left|\frac{E(x(t),y(t))}{|\vec{r}(t)-\vec{r}(t_0)|}\right| \left|\frac{|\vec{r}(t)-\vec{r}(t_0)|}{t-t_0}\right| =$$

$$\lim_{t\to t_0} \left|\frac{E(x(t),y(t))}{|\vec{r}(t)-\vec{r}(t_0)|}\right| |\vec{r}\,'(t_0)| =$$

$$\lim_{(x,y)\to(x_0,y_0)} \left|\frac{E(x,y)}{|\langle x,y\rangle - \langle x_0,y_0\rangle|}\right| |\vec{r}\,'(t_0)| = 0(|\vec{r}\,'(t_0)|) = 0.$$

Therefore

$$\boxed{\frac{d}{dt}\left(f(\vec{r}(t_0))\right) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)}$$