Chapter 1

Introduction

In this chapter, we give an overview of the topics contained in this book. We follow the historical arc of quaternion algebras and see in broad stroke how they have impacted the development of many areas of mathematics. This account is selective and is mostly culled from existing historical surveys; two very nice surveys of quaternion algebras and their impact on the development of algebra are those by Lam [Lam03] and Lewis [Lew2006].

1.1 Hamilton's quaternions

In perhaps the most famous act of mathematical vandalism, on October 16, 1843, Sir William Rowan Hamilton carved the following equations into the Brougham Bridge (now Broomebridge) in Dublin:

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.1.1}$$

His discovery of these multiplication laws was a defining moment in the history of algebra.

For at least ten years, Hamilton had been attempting to model three-dimensional space with a structure like the complex numbers, whose addition and multiplication model two-dimensional space. Just like the complex numbers had a "real" and "imaginary" part, so too did Hamilton hope to find an algebraic system whose elements had a "real" and two-dimensional "imaginary" part. His son William Edward Hamilton, while still very young, would pester his father [Ham67, p. xv]: "Well, papa, can you multiply triplets?" To which Hamilton would reply, with a sad shake of the head, "No, I can only add and subtract them." (For a history of the "multiplying triplets" problem—the nonexistence of division algebra over the reals of dimension 3—see May [May66, p. 290].)

Then, on this dramatic day in 1843, Hamilton's had a flash of insight [Ham67, p. xx–xxvi]:

> On the 16th day of [October]—which happened to be a Monday, and a Council day of the Royal Irish Academy—I was walking in to attend and

Figure 1.1: Sir William Rowan Hamilton (1805–1865)

> preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse—unphilosophical as it may have been—to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely,
>
> $$i^2 = j^2 = k^2 = ijk = -1$$
>
> which contains the Solution of the Problem.

In this moment, Hamilton realized that he needed a fourth dimension, and so he coined the term *quaternions* for the real space spanned by the elements $1, i, j, k$, subject to his multiplication laws (1.1.1). Later, he presented this theory to the Royal Irish Academy in a paper entitled "On a new Species of Imaginary Quantities connected with a theory of Quaternions" [Ham1843]. Although his carvings have long since worn away, a plaque on the bridge now commemorates this historically significant event. This magnificent story remains in the popular consciousness, and to commemorate Hamilton's discovery of the quaternions, there is an annual "Hamilton walk" in Dublin [ÓCa2010].

For more on the history of Hamilton's discovery, see the extensive and detailed accounts of Dickson [Dic19] and van der Waerden [vdW76]. There are three main biographies written about the life of William Rowan Hamilton, a man sometimes referred to as "Ireland's greatest mathematician", by Graves [Grav1882, Grav1885, Grav1889] in three volumes, Hankins [Han80], and O'Donnell [O'Do83]. Numerous other shorter biographies have been written [Lanc67, ÓCa2000].

Although Hamilton was undoubtedly responsible for advancing the theory of quaternion algebras, there are several precursors to his discovery that bear mentioning. First, the quaternion multiplication laws are already implicitly present in the four-square identity of Leonhard Euler:

$$\begin{aligned}(a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2) = c_1^2+c_2^2+c_3^2+c_4^2 = \\ (a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2+(a_1b_2+a_2b_1+a_3b_4-a_4b_3)^2 \\ +(a_1b_3-a_2b_4+a_3b_1+a_4b_2)^2+(a_1b_4+a_2b_3-a_3b_2+a_4b_1)^2.\end{aligned} \tag{1.1.2}$$

Indeed, the multiplication law in the quaternion reads precisely

$$(a_1+a_2i+a_3j+a_4k)(b_1+b_2i+b_3j+b_4k) = c_1+c_2i+c_3j+c_4k.$$

It was perhaps Carl Friedrich Gauss who first observed this connection. In a note dated around 1819 [Gau00], he interpreted the formula (1.1.2) as a way of composing real quadruples: to the quadruples (a_1,a_2,a_3,a_4) and (b_1,b_2,b_3,b_4) in $\mathbb{R}^4$, he defined the composite tuple (c_1,c_2,c_3,c_4) and noted the noncommutativity of this operation. Gauss elected not to publish these findings (as he chose not to do with many of his discoveries). In letters to De Morgan [Grav1885, Grav1889, p. 330, p. 490], Hamilton attacks the allegation that Gauss had discovered quaternions first. Finally, Olinde Rodrigues (1795–1851) (of the *Rodrigues formula* for Legendre polynomials) gave a formula for the angle and axis of a rotation in $\mathbb{R}^3$ obtained from two successive rotations—essentially giving a different parametrization of the quaternions—but had left mathematics for banking long before the publication of his paper [Rod1840]. The story of Rodrigues and the quaternions is given by Altmann [Alt89] and Pujol [Puj2012] and the fuller story of his life by Altmann–Ortiz [AO05].

In any case, the quaternions consumed the rest of Hamilton's academic life and resulted in the publication of two treatises [Ham1853, Ham1866] (see also the review [Ham1899]). Hamilton's writing over these years became increasingly obscure, and many found his books to be impenetrable. Nevertheless, many physicists used quaternions extensively and for a long time in the mid-19th century, quaternions were an essential notion in physics. Hamilton endeavored to set quaternions as the standard notion for vector operations in physics as an alternative to the more general dot product the *laws of i, j, k agree* with usual and *algebraic laws:* namely, in the *Associative Property of Multiplication;* or in the property that the new symbols always obey the *associative formula* (comp. 9),

$$\iota . \kappa\lambda = \iota\kappa . \lambda,$$

whichever of them may be substituted for ι, for κ, and for λ; in virtue of which equality of values we may *omit the point*, in any such symbol of a *ternary product* (whether of equal or of unequal factors), and write it simply as $\iota\kappa\lambda$. In particular we have thus,

$$i.jk = i.i = i^2 = -1; \qquad ij.k = k.k = k^2 = -1;$$

or briefly,

$$ijk = -1.$$

We may, therefore, by 182, establish the following important *Formula:*

$$i^2 = j^2 = k^2 = ijk = -1; \qquad \text{(A)}$$

to which we shall occasionally refer, as to "Formula A," and which we shall find to contain (virtually) *all the laws of the symbols ijk*, and therefore to be a *sufficient symbolical basis* for the whole *Calculus of Quaternions:** because it will be shown that *every quaternion can be reduced to the Quadrinomial Form,*

$$q = w + ix + jy + kz,$$

where w, x, y, z compose a *system of four scalars*, while i, j, k are the same *three right versors* as above.

(1.) A direct proof of the equation, $ijk = -1$, may be derived from the definitions of the symbols in Art. 181. In fact, we have only to remember that those definitions were seen to give,

---

* This formula (A) was accordingly made the *basis* of that Calculus in the first communication on the subject, by the present writer, to the Royal Irish Academy in 1843; and the letters, *i, j, k*, continued to be, for some time, the *only peculiar symbols* of the calculus in question. But it was gradually found to be useful to incorporate with these a few other *notations* (such as K and U, &c.), for representing *Operations on Quaternions*. It was also thought to be instructive to establish the *principles* of that Calculus, on a more *geometrical* (or less exclusively *symbolical*) *foundation* than at first; which was accordingly afterwards done, in the volume entitled: *Lectures on Quaternions* (Dublin, 1853); and is again attempted in the present work, although with many differences in the adopted *plan* of exposition, and in the *applications* brought forward, or suppressed.

Figure 1.2: A page from Hamilton's *Elements of quaternions*

and cross product introduced in 1881 by Willard Gibbs (1839–1903), building on remarkable but largely ignored work of Hermann Grassmann (1809–1877) [Gras1862]. The two are related by the beautiful equality

$$vw = v \cdot w + v \times w \tag{1.1.3}$$

for $v, w \in \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$, relating quaternionic multiplication to dot and cross products. This rivalry between physical notation flared into a war in the latter part of the 19th century between the 'quaternionists' and the 'vectorists', and for some the preference of one system versus the other became an almost partisan split. On the side of quaternions, James Clerk Maxwell (1831–1879) (responsible for the equations which describe electromagnetic fields) wrote [Max1869, p. 226]:

> The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared, for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.

And Peter Tait (1831–1901), one of Hamilton's students, wrote in 1890 [Tai1890]:

> Even Prof. Willard Gibbs must be ranked as one the retarders of quaternions progress, in virtue of his pamphlet on *Vector Analysis*, a sort of hermaphrodite monster, compounded of the notation of Hamilton and Grassman.

On the vectorist side, Lord Kelvin (a.k.a. William Thomson, who formulated the laws of thermodynamics), said in an 1892 letter to R. B. Hayward about his textbook in algebra (quoted in Thompson [Tho10, p. 1070]):

> Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell.

Ultimately, the superiority of vector notation carried the day, and only certain useful fragments of Hamilton's quaternionic notation (e.g., the "right-hand rule" $i \times j = k$ in multivariable calculus) remain in modern usage. For more on the history of quaternionic and vector calculus, see Crowe [Cro64] and Simons [Sim2010].

The debut of the quaternions by Hamilton was met with some resistance in the mathematical world: it proposed a system of "numbers" that did not satisfy the usual commutative rule of multiplication. Quaternions predated the notion of matrices, introduced in 1855 by Arthur Cayley (1821–1895). Hamilton's bold proposal of a noncommutative multiplication law was the harbinger of an array of algebraic structures. In the words of J.J. Sylvester [Syl1883, pp. 271–272]:

> In Quaternions (which, as will presently be seen, are but the simplest order of matrices viewed under a particular aspect) the example had been given of Algebra released from the yoke of the commutative principle of multiplication—an emancipation somewhat akin to Lobachevsky's of

> Geometry from Euclid's noted empirical axiom; and later on, the Peirces, father and son (but subsequently to 1858) had prefigured the universalization of Hamilton's theory, and had emitted an opinion to the effect that probably all systems of algebraical symbols subject to the associative law of multiplication would be eventually found to be identical with linear transformations of schemata susceptible of matriculate representation.

Indeed, with the introduction of the quaternions the floodgates of algebraic possibilities had been opened. See Happel [Hap80] for the early development of algebra following Hamilton's quaternions.

1.2 Algebra after the quaternions

Soon after he discovered his quaternions, Hamilton sent a letter [Ham1844] describing them to his friend John T. Graves (1806–1870). Graves replied on October 26, 1843, with his complements, but added:

> There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties. ... If with your alchemy you can make three pounds of gold, why should you stop there?

Following through on this invitation, on December 26, 1843, Graves wrote to Hamilton that he had successfully generalized the quaternions to the "octaves", now called *octonions* $\mathbb{O}$, an algebra in eight dimensions, with which he was able to prove that the product of two sums of eight perfect squares is another sum of eight perfect squares, a formula generalizing (1.1.2). In fact, Hamilton first invented the term *associative* in 1844, around the time of his correspondence with Graves. Unfortunately for Graves, the octonions were discovered independently and published already in 1845 by Cayley [Cay1845], who often is credited for their discovery. (Even worse, the eight squares identity was also previously discovered by C. F. Degen.) For a more complete account of this story and the relationships between quaternions and octonions, see the survey article by Baez [Bae02], the article by van der Blij [vdB60], and the delightful book by Conway–Smith [CS03].

Cayley was able to reinterpret the quaternions as arising from a *doubling process*, also called the *Cayley–Dickson construction*, which starting from $\mathbb{R}$ produces $\mathbb{C}$ then $\mathbb{H}$ then $\mathbb{O}$, taking the ordered, commutative, associative algebra $\mathbb{R}$ and progressively deleting one adjective at a time. So algebras were first studied over the real and complex numbers and were accordingly called *hypercomplex numbers* in the late 19th and early 20th century. And this theory flourished. In 1878, Ferdinand Frobenius (1849–1917) proved that the only finite-dimensional division associative algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ [Fro1878]. (This result was also proven independently by C.S. Peirce, the son of Benjamin Peirce, below.) Much later, work by topologists culminated in the theorem of Bott–Milnor [BM58] and Kervaire [Ker58]: the only finite-dimensional division (not-necessarily-associative) algebras have dimensions $1, 2, 4, 8$. As a consequence, the $(n-1)$-dimensional sphere

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\|^2 = 1\}$$

has a trivial tangent bundle if and only if $n = 1, 2, 4, 8$.

In another attempt to seek a generalization of the quaternions to higher dimension, William Clifford (1845–1879) developed a way to build algebras from quadratic forms in 1876 [Cli1878]. Clifford constructed what we now call a *Clifford algebra* associated to $V = \mathbb{R}^n$; it is an algebra of dimension 2^n containing V with multiplication induced from the relation $x^2 = -\|x\|^2$ for all $x \in V$. We have $C(\mathbb{R}^1) = \mathbb{C}$ and $C(\mathbb{R}^2) = \mathbb{H}$, so the Hamilton quaternions arise as a Clifford algebra, but $C(\mathbb{R}^3)$ is not the octonions. Nevertheless, the theory of Clifford algebras is tightly connected to the theory of normed division algebras. For more on the history of Clifford algebras, see Diek–Kantowski [DK95].

The study of division algebras gradually evolved, including work by Benjamin Peirce [Pei1882] originating from 1870 on *linear associative algebra*; therein, he provides a decomposition of an algebra relative to an idempotent. The notion of a *simple* algebra had been found and developed around this time by Élie Cartan (1869–1951). But it was Joseph Henry Maclagan Wedderburn (1882–1948) who was the first to find meaning in the structure of simple algebras over an arbitrary field, in many ways leading the way forward. The jewel of his 1908 paper [Wed08] is still foundational in the structure theory of algebras: a simple algebra (finite-dimensional over a field) is isomorphic to a matrix ring over a division ring. Wedderburn also proved that a finite division ring is a field, a result that like his structure theorem has inspired much mathematics. For more on the legacy of Wedderburn, see Artin [Art50].

Around this time, other types of algebras over the real numbers were also being investigated, the most significant of which were Lie algebras. In the seminal work of Sophus Lie (1842–1899), group actions on manifolds were understood by looking at this action infinitessimally; one thereby obtains a *Lie algebra* of vector fields that determines the local group action. The simplest nontrivial example of a Lie algebra is the cross product of two vectors, related to quaternion multiplication in (1.1.3): it defines, in fact, a binary operation on $\mathbb{R}^3$, but now

$$i \times i = j \times j = k \times k = 0.$$

The Lie algebra “linearizes” the group action and is therefore more accessible. Wilhelm Killing (1847–1923) initiated the study of the classification of Lie algebras in a series of papers [Kil1888], and this work was completed by Cartan. For more on this story, see Hawkins [Haw00].

The first definition of an algebra over an arbitrary field seems to have been given by Leonard E. Dickson (1874–1954) [Dic03] (even though at first he still called the resulting object a *system of complex numbers* and later adopting the name *linear algebra*). In the early 1900s, Dickson developed this theory further and in particular was the first to consider quaternion algebras over a general field. First, he considered algebras in which every element satisfies a quadratic equation [Dic12], leading to multiplication laws for what he later called a *generalized quaternion algebra* [Dic14, Dic23].

Today, we no longer employ the adjective "generalized"—over fields other than $\mathbb{R}$, there is no reason to privilege the Hamiltonians—and we can reinterpret this vein of Dickson's work as showing that every 4-dimensional central simple algebra is a quaternion algebra (over a field F with $\operatorname{char} F \neq 2$).

At this time, Dickson [Dic19] (giving also a complete history) wrote on earlier work of Hurwitz (1859–1919) from 1888 [Hur1888], who asked for generalizations of the composition laws arising from sum of squares laws like that of Euler (1.1.2) for four squares and Cayley for eight squares: for which n does there exist an identity

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) = c_1^2 + \cdots + c_n^2$$

with each c_i bilinear in the variables a and b? He showed they only exist for $n = 1, 2, 4, 8$ variables (so in particular, there is no formula expressing the product of two sums of 16 squares as the sum of 16 squares), the result being tied back to his theory of algebras.

Biquaternion (Albert) algebras. A. Adrian Albert. [[Finish; only outline is written.]]

Class field theory Hasse principle (1920s), class field theory, Noether, arithmetic of hypercomplex number systems. Cyclic algebras, cyclic cross product.

As "twisted forms" of 2×2-matrices, quaternion algebras in many ways are like "noncommutative quadratic field extensions", and just as the quadratic fields $\mathbb{Q}(\sqrt{d})$ are wonderously rich, so too are their noncommutative analogues. In this way, quaternion algebras provide a natural place to do noncommutative algebraic number theory. A more general study would look at central simple algebras (see Reiner).

Fuchsian groups The quotient gives rise to a *Riemann surface*. Riemann.

Hypergeometric functions also give examples. Fuchs and his differential equations.

After all, how do you get discrete groups? Start with real matrices, go to rational matrices, then to integral matrices, then make a group. Allow yourself entries in a number field, consider the algebra generated, take integral elements, make the group. When is this discrete? Something like 4-dimensional object gives you quaternion algebras.

Modular forms. The basic example being the group $\mathrm{SL}_2(\mathbb{Z})$. Quaternion algebras give rise therefore to objects of interest in geometry and low-dimensional topology. Classical modular forms.

discovered by Deuring.

Especially Jacobi and the sums of 4 squares, something that also can be seen using quaternion algebras. How often is an integer a sum of squares, or more generally, represented by a quadratic form in 4 variables? The generating function is a modular form.

Automorphic forms. Then discovery by Poincaré "when he was walking on a cliff," apparently in 1886, as he reminisced in his *Science et Méthode*. Holomorphic (complex analytic) functions that are invariant with respect to these groups are very interesting to study ("automorphic functions"). Set of matrices that preserve a quadratic form.

Soon after, this was followed by Fricke and Klein, who were interested in subgroups of $\mathrm{PGL}_2(\mathbb{R})$ that act discretely on the upper half-plane, such as the group generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{2} & 1+\sqrt{3} \\ 1-\sqrt{3} & \sqrt{2} \end{pmatrix}.$$

They still used the language of quadratic forms? In this way, quaternion algebras are useful in group theory.

Then other groups, Hilbert modular forms.

1.3 Modern theory

Composition laws Picking up again, work of Brandt.

Hecke operators, the basis problem, and the trace formula

Then Eichler: theory of Hecke operators in the 1950s. Selberg and trace formula. Basis problem.

Modularity and elliptic curves

In the theory of modular forms, the Hecke operators (Petersson, Maass) determine the coefficients of modular forms and Dirichlet series. These operators permit a vast generalization, replacing modular groups by other groups: the group of units in a central simple algebra or of certain quadratic forms; these essentially coincide for quaternion algebras, which also speaks back to the modular case. One can compute traces of these operators quite explicitly.

Work of Shimura: find examples of zeta functions that could be given. Theory of complex multiplication and modularity of elliptic curves. Galois representations

Pizer [Piz80, §§1–2] gives a concise introduction to the "algebraic and arithmetic of quaternion algebras" over $\mathbb{Q}$.

Abelian varieties. Quaternion algebras arise also as the endomorphism rings of elliptic curves, and indeed they are the only noncommutative endomorphism algebras of simple abelian varieties over fields by Albert's classification. So that justifies there study already. The Rosati involution figures prominently in this classification.

Algebras with involution

Composition algebras. Algebras with involutions: Knus, etc. Connects back to Lie theory.

Riemannian manifolds

Back to Riemann surfaces. Vignéras.

Arithmetic groups

Three-dimensional groups, arithmetic, some results.

Algorithmic aspects. Computations and algorithms can be done; this gives modular symbols, Brandt matrices, and their generalizations.

Today, quaternions have seen a revival in computer modeling and animation as well as in attitude control of aircraft and spacecraft [Han06]. A rotation in $\mathbb{R}^3$ about an axis through the origin can be represented by a 3×3 orthogonal matrix with determinant 1. However, the matrix representation is redundant, as there are only four degrees of freedom in such a rotation (three for the axis and one for the angle). Moreover, to compose two rotations requires the product of the two corresponding matrices, which requires 27 multiplications and 18 additions in $\mathbb{R}$. Quaternions, on the other hand, represent this rotation with a 4-tuple, and multiplication of two quaternions takes only 16 multiplications and 12 additions in $\mathbb{R}$. [[What about Euler angles?]]

In physics, quaternions yield elegant expression for the Lorentz transformations, the basis of the modern theory of relativity. There has been renewed interest by topologists in understanding quaternionic manifolds and by physicists who seek a quaternionic quantum physics, and some physicists still hope they will obtain a deeper understanding of physical principles in terms of quaternions. [[Ways to visualize the spin group [HFK94].]] And so although much of Hamilton's quaternionic physics fell out of favor long ago, we have somehow come full circle in our elongated historical arc. The enduring role of quaternion algebras as a progenitor of a vast range of mathematics promises a rewarding ride for years to come.

Exercises

1.1. Hamilton originally sought an associative multiplication law on

$$B = \mathbb{R} + \mathbb{R}i + \mathbb{R}j \cong \mathbb{R}^3$$

where $i^2 = -1$, so in particular $\mathbb{C} \subset B$. Show that no such multiplication law can exist.

1.2. Hamilton sought a multiplication $* : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ that preserves length:

$$\|v\|^2 \cdot \|w\|^2 = \|v * w\|^2$$

for $v, w \in \mathbb{R}^3$. Expanding out in terms of coordinates, such a multiplication would imply that the product of the sum of three squares over $\mathbb{R}$ is again the sum of three squares in $\mathbb{R}$. (Such a law holds for the sum of two squares, corresponding to the multiplication law in $\mathbb{R}^2 \cong \mathbb{C}$: we have

$$(x^2 + y^2)(u^2 + v^2) = (xu - yv)^2 + (xv + yu)^2.)$$

However, show that such a formula for three squares is impossible, as it would imply an identity in the polynomial ring in 6 variables over $\mathbb{Z}$. *[Hint: Find a natural number that is the product of two sums of three squares which is not itself the sum of three squares.]*

1.3. Show that there is no way to give $\mathbb{R}^3$ the structure of a ring (with 1) in which multiplication distributes over scalar multiplication by $\mathbb{R}$ and every nonzero element has a (two-sided) inverse, as follows.

a) Suppose otherwise, and $\mathbb{R}^3 = D$ is equipped with a multiplication law. Show that every $\alpha \in D$ satisfies a polynomial of degree at most 3 with coefficients in $\mathbb{R}$.

b) By consideration of irreducible factors, show that every $\alpha \in D$ satisfies a (minimal) polynomial of degree 1.

c) Derive a contradiction from the fact that every nonzero element has a (two-sided) inverse.