Chapter 29

Classical modular group

In this section, we introduce the classical modular group $\text{SL}_2(\mathbb{Z})$, examine the hyperbolic quotient in detail, and we discuss some arithmetic applications.

29.1 The fundamental domain

The classical modular group is the group

$$\text{PSL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} / \{\pm 1\} \leq \text{PSL}_2(\mathbb{R}).$$

The group $\text{PSL}_2(\mathbb{Z})$ acts faithfully on the upper half-plane $\mathbb{H}^2$ by linear fractional transformations, and moreover when equipped with the hyperbolic metric this action is by orientation-preserving isometries.

Since $\mathbb{Z} \subseteq \mathbb{R}$ is discrete, so too is $\text{SL}_2(\mathbb{Z}) \subseteq \text{M}_2(\mathbb{Z}) \subseteq \text{M}_2(\mathbb{R})$ discrete, and therefore $\text{PSL}_2(\mathbb{Z}) \leq \text{PSL}_2(\mathbb{R})$ is a Fuchsian group. Therefore, by Proposition 28.5.2, the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$ is wandering and proper. [[We actually prove this below by direct computation, so maybe this chapter should move.]]

Our first order of business is to try to understand the structure of the classical modular group in terms of this action. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}).$$

Then $Sz = -1/z$ for $z \in \mathbb{H}^2$, so $S$ maps the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ to itself, fixing the point $z = i$; and $Tz = z + 1$ for $z \in \mathbb{H}^2$ acts by translation. We compute that $S^2 = 1$ (in $\text{PSL}_2(\mathbb{Z})$) and

$$ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

so $(ST)^3 = 1$. 329
In a moment, we will see that $\text{PSL}_2(\mathbb{Z})$ is generated by $S, T$, with a minimal set of relations given by $S^2 = (ST)^3 = 1$. To do so, we examine a fundamental set (cf. Definition 28.1.13) for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$, as follows. Let

$$\Xi = \{ z \in \mathbb{H}^2 : |\text{Re} z| \leq 1 \text{ and } |z| \geq 1 \}.$$ 

The set $\Xi$ is a hyperbolic triangle with vertices at $\omega = (-1 + \sqrt{-3})/2$ and $-\omega^2 = (1 + \sqrt{-3})/2$ and $\infty$. Its translates by words in $S, T$ looks as follows:

By the Gauss–Bonnet Theorem 27.5.5 (or Exercise 29.1), the hyperbolic area of $\Xi$ is $\pi - 2(\pi/3) = \pi/3$. The elements $S, T$ act on the edges of this triangle as follows:

In the unit disc, the triangle $\Xi$ looks like this:
The following three lemmas describe the relationship of the set \( \Pi \) to \( \Gamma \).

**Lemma 29.1.1.** For all \( z \in H^2 \), there exists a word \( \gamma \in \langle S, T \rangle \) such that \( \gamma z \in \Pi \).

**Proof.** In fact, we can determine such a word algorithmically. First, we translate \( z \) so that \( |\text{Re } z| \leq 1/2 \). If \( |z| \geq 1 \), we are done; otherwise, if \( |z| < 1 \), then

\[
\text{Im} \left( \frac{-1}{z} \right) = \frac{\text{Im } z}{|z|^2} > \text{Im } z.
\]

We then repeat this process, obtaining a sequence of elements \( z = z_1, z_2, \ldots \) with \( \text{Im } z_1 < \text{Im } z_2 < \ldots \). We claim that this process terminates after finitely many steps. Indeed, we have

\[
\text{Im}(gz) = \frac{\text{Im } z}{|cz + d|^2}, \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}),
\]

and the number of \( c, d \in \mathbb{Z} \) such that \( |cz + d| < 1 \) is finite: the set \( \mathbb{Z} + \mathbb{Z}z \subseteq \mathbb{C} \) is a lattice, so there are only finitely many elements of bounded norm. Upon termination, we have found a word \( \gamma \) in \( S, T \) such that \( \gamma z \in \Pi \). \( \square \)

The procedure described in the proof of Lemma 29.1.1 is called a reduction algorithm.

**Lemma 29.1.3.** Let \( z, z' \in \Pi \), and suppose \( z \in \text{int}(\Pi) \) lies in the interior of \( \Pi \). If \( z' = \gamma z \) with \( \gamma \in \Gamma \), then \( \gamma = 1 \) and \( z = z' \).

**Proof.** Let \( z' = \gamma z \) with \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). We have \( \text{Im } z' = (\text{Im } z)/(|cz + d|^2) \). First suppose that \( \text{Im } z' \geq \text{Im } z \); then

\[
|cz + d|^2 = (c \text{Re } z + d)^2 + c^2(\text{Im } z)^2 \leq 1.
\]

Since \( \text{Im } z > \text{Im } \omega = \sqrt{3}/2 \), from (29.1.4) we conclude that \( c^2 \leq 4/3 \) so \( |c| \leq 1 \). If \( c = 0 \) then \( ad - bc = ad = 1 \) so \( a = d = \pm 1 \), and then \( z' = \gamma z = z \pm b \), which immediately implies \( b = 0 \) so \( \gamma = 1 \) as claimed. If instead \( |c| = 1 \), then the conditions
(c Re z + d)^2 \leq 1 - (\text{Im } z)^2 \leq 1 - 3/4 = 1/4 \quad \text{and} \quad |\text{Re } z| < 1/2

together imply d = 0; but then |cz + d| = |z| \leq 1, and since z \in \text{int}(\mathbb{H}) we have |z| > 1, a contradiction.

If instead Im z' < Im z, we interchange the roles of z, z' and have strict inequality in \((29.1.4)\); by the same argument and the weaker inequality |Re z| \leq 1/2, we then obtain |z| < 1, a contradiction. \hfill \Box

Lemma 29.1.5. The elements S, T generate \( \Gamma = \text{PSL}_2(\mathbb{Z}) \).

Proof. Let \( z_0 = 2i \in \text{int}(\mathbb{H}) \). Let \( \gamma \in \Gamma \), and let \( z' = \gamma z \). By Lemma 29.1.1, there exists \( \gamma' \) a word in S, T such that \( \gamma' z' \in \mathbb{H} \). By Lemma 29.1.3, we have \( \gamma' z' = (\gamma' \gamma) z = z \), so \( \gamma' \gamma = 1 \) and \( \gamma = \gamma' \in \langle S, T \rangle \).

Although we have worked in \( \text{PSL}_2(\mathbb{Z}) \) throughout, it follows from Lemma 29.1.5 that the matrices S, T also generate \( \text{SL}_2(\mathbb{Z}) \), since \( S^2 = -1 \).

The following corollary is an immediate consequence of Lemmas 29.1.1 and 29.1.3.

Corollary 29.1.6. The set \( \mathbb{H} \) is a fundamental set for \( \text{PSL}_2(\mathbb{Z}) \) \( \cap \mathbb{H}^2 \).

29.1.7. If \( z \in \mathbb{H} \) has \( \text{Stab}_\Gamma(z) \neq \{1\} \), then we claim that one of the following holds:

(i) \( z = i \), and \( \text{Stab}_\Gamma(i) = \langle S \rangle \simeq \mathbb{Z}/2\mathbb{Z} \);

(ii) \( z = \omega \), and \( \text{Stab}_\Gamma(\omega) = \langle ST \rangle \simeq \mathbb{Z}/3\mathbb{Z} \); or

(iii) \( z = -\omega^2 \), and \( \text{Stab}_\Gamma(-\omega^2) = \langle TS \rangle = T \text{Stab}_\Gamma(\omega)T^{-1} \).

Indeed, let \( \gamma z = z \) with \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( \gamma \neq 1 \). Then \( cz^2 + (d-a)z - b = 0 \), so \( c \neq 0 \) and

\[
   z = \frac{(a-d) + \sqrt{D}}{2c}
\]

where \( D = \text{Tr}(\gamma)^2 - 4 \in \mathbb{Z}_{<0} \).

Thus we have \( D = -4 \) or \( D = -3 \). In either case, since \( z \in \mathbb{H} \) we have \( \text{Im } z \geq \sqrt{3}/2 \), we must have \( c = \pm 1 \), and replacing \( \gamma \leftarrow -\gamma \) we may take \( c = 1 \). If \( D = -4 \), then \( \text{Tr}(\gamma) = a + d = 0 \) so \( z = a + i \), and so \( a = 0 = d \) and \( c = 1 = -b \), i.e., \( z = i \) and we have case (i). If the discriminant is \(-3\), then a similar argument gives \( z = ((a \pm 1) + \sqrt{-3})/2 \) so \( a = 0 \), and we are in cases (ii) or (iii).

Let \( Y = \Gamma \\setminus \mathbb{H}^2 \). Gluing together the fundamental set, we obtain a homeomorphism

\[
   Y = \Gamma \\setminus \mathbb{H}^2 \simeq \mathbb{P}^1(\mathbb{C}) - \{\infty\} \simeq \mathbb{C}.
\]
29.2 Binary quadratic forms

We pause to give an application to quadratic forms and class groups, following Gauss. An \((\text{integral})\) binary quadratic form, abbreviated in this section to simply form, is an expression \(Q(x, y) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y]\); we define the discriminant of a form \(Q\) to be \(d = \text{disc}(Q) = b^2 - 4ac\). A form \(Q\) is primitive if \(\gcd(a, b, c) = 1\), and \(Q\) is positive definite if \(Q(x, y) > 0\) for all nonzero \((x, y) \in \mathbb{R}^2\); after completing the square, we see that a form is positive definite if and only if \(a > 0\) and \(d < 0\). For a negative discriminant \(d < 0\), let

\[Q_d = \{Q(x, y) = ax^2 + bxy + cy^2 : a > 0, \text{disc}(Q) = d\}\]

be the set of primitive, positive definite forms of discriminant \(d\). The group \(\Gamma\) acts on \(Q_d\) the right by change of variable: for \(\gamma \in \Gamma\), we define \((Q^\gamma)(x, y) = Q((x, y)\gamma)\), and verify that \(\text{disc}(Q^\gamma) = \text{disc}(Q) = d\). We say that \(Q, Q'\) are equivalent if \(Q' = Q^\gamma\) for some \(\gamma \in \Gamma\).

We claim that the number of equivalence classes \(h(d) = \#Q_d/\Gamma\) is finite. Indeed, to any \(Q \in Q_d\), we associate the unique root

\[z_Q = \frac{-b + \sqrt{|d|i}}{2a} \in \mathbb{H}\]

of \(Q(z, 1) = 0\). Then \(z_{Q^\gamma} = \gamma^{-1}(z)\) for \(\gamma \in \Gamma\). Therefore, by the reduction theory of the previous section, we can replace \(Q\) up to equivalence by a form such that \(z_Q \in \mathbb{H}\).
If we further insist that $\Re z < 1/2$ and $\Re z < 0$ if $|z| = 1$, then this representative is unique.

Thus

$$-\frac{1}{2} \leq \Re z_Q = -\frac{b}{2a} < \frac{1}{2}$$

so $-a < b \leq a$, or equivalently,

$$|b| \leq a \text{ and } (b \geq 0 \text{ if } |b| = a);$$

and

$$|z_Q| = \frac{b^2 - d}{4a^2} = \frac{c}{a} \geq 1$$

so $a \leq c$ and $b \geq 0$ if equality holds. In sum, every positive definite form $Q$ is equivalent to a $(\text{SL}_2(\mathbb{Z})$-)reduced form satisfying

$$|b| \leq a \leq c \quad \text{with } b \geq 0 \quad \text{if } |b| = a \text{ or } a = c.$$  

We now show that there are only finitely many reduced forms with given discriminant $d < 0$, i.e., that $h(d) < \infty$. The inequalities $|b| \leq a \leq c$ imply that

$$|d| = 4ac - b^2 \geq 3a^2,$$

so $a \leq \sqrt{|d|/3}$ and $|b| \leq a$, so there are only finitely many possibilities for $a, b$; and then $c = (b^2 - d)/(4a)$ is determined. This gives an efficient method to compute the set $\mathbb{Q}_d/\Gamma$ efficiently.

Let $S = \mathbb{Z} \oplus \mathbb{Z}[(d + \sqrt{d})/2] \subset K = \mathbb{Q}(\sqrt{d})$ be the quadratic ring of discriminant $d < 0$. Let Pic$(S)$ be the group of invertible fractional ideals of $S$ modulo principal ideals. Then there is a bijection

$$\mathbb{Q}_d/\Gamma \leftrightarrow \text{Pic}(S)$$

$$[ax^2 + bxy + cy^2] \mapsto [a] = \left[ a, \frac{-b + \sqrt{d}}{2} \right]$$

(Exercise 29.7). So in the same stroke, we have proven the finiteness of the class number $\# \text{Pic}(S) < \infty$. 


29.3 Functions on lattices, and complex tori

In this section, we pursue an interpretation of the quotient $\Gamma \backslash \mathbb{H}^2$ with $\Gamma = \text{PSL}_2(\mathbb{Z})$ as a moduli space of lattices.

A lattice $\Lambda \subset \mathbb{C}$ is a subgroup $\Lambda = \mathbb{Z}z_1 + \mathbb{Z}z_2$ with $z_1, z_2$ linearly independent over $\mathbb{R}$; the elements $z_1, z_2$ are a basis for $\Lambda$.

Two lattices $\Lambda, \Lambda'$ are homothetic if there exists $u \in \mathbb{C}^\times$ such that $\Lambda' = u\Lambda$, and we write $\Lambda \sim \Lambda'$. Let $\Lambda = \mathbb{Z}z_1 + \mathbb{Z}z_2$ be a lattice. Then without loss of generality (interchanging $z_1, z_2$), we may assume $\text{Im}(z_2/z_1) > 0$, and then we call $z_1, z_2$ an oriented basis. Then we have the homothety

$$\Lambda \sim z_1^{-1}\Lambda = \mathbb{Z} + \mathbb{Z}\tau$$

where $\tau = z_2/z_1 \in \mathbb{H}^2$; this representative of the homothety class is unique up to the action of $\Gamma$, as follows.

**Lemma 29.3.1.** We have $\mathbb{Z} + \mathbb{Z}\tau \sim \mathbb{Z} + \mathbb{Z}\tau'$ if and only if $\Gamma\tau = \Gamma\tau'$.

**Proof.** For $\tau, \tau' \in \mathbb{C} - \mathbb{R}$, we have $\mathbb{Z} + \mathbb{Z}\tau \sim \mathbb{Z} + \mathbb{Z}\tau'$ if and only if there exists $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$ and $u \in \mathbb{C}^\times$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = u \begin{pmatrix} \tau' \\ 1 \end{pmatrix}$$

so $u(a\tau + b) = \tau'$ and $u(c\tau + d) = 1$. Eliminating $u$ gives equivalently

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$  

Finally, if $\tau, \tau' \in \mathbb{H}^2$ then $g \in \text{PSL}_2(\mathbb{Z})$, and conversely the action of $\Gamma$ preserves $\mathbb{H}^2$; so since $g$ is well-defined as an element of $\Gamma = \text{PSL}_2(\mathbb{Z})$, we have the result. \(\square\)

Therefore, we have a bijection

$$Y = \Gamma \backslash \mathbb{H}^2 \to \{ \Lambda \subset \mathbb{C} \text{ lattice} \} / \sim$$

$$\Gamma\tau \mapsto [\mathbb{Z} + \mathbb{Z}\tau];$$

that is to say, $Y = \Gamma \backslash \mathbb{H}^2$ parametrizes complex lattices up to homothety.

To a lattice $\Lambda$, we associate the complex torus $\mathbb{C}/\Lambda$ (of rank 1); two such tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic as Riemann surfaces if and only if $\Lambda \sim \Lambda'$. Therefore, the space $Y$ also parametrizes complex tori.

In particular, like $Y$, the set of homothety classes has a natural structure of a Riemann surface and we seek now to make this explicit. There are natural functions on lattices that allow us to go beyond a bijection and realize the complex structure on $Y$ explicitly.

Let $\Lambda \subset \mathbb{C}$ be a lattice. To write down complex moduli, it is natural to average over $\Lambda$ in a convergent way, as follows.
Definition 29.3.2. The Eisenstein series of weight \( k \in \mathbb{Z}_{>2} \) for \( \Lambda \) to be

\[
G_k(\Lambda) = \frac{1}{2} \sum_{\lambda \in \Lambda, \lambda \neq 0} \frac{1}{\lambda^k}.
\]

If \( k \) is odd, then \( G_k(\Lambda) = 0 \) identically, so let \( k \in 2\mathbb{Z}_{>2} \).

Lemma 29.3.3. The series \( G_k(\Lambda) \) converges absolutely.

Proof. Up to homothety (which does not affect convergence), we may assume \( \Lambda = \mathbb{Z} + \mathbb{Z}\tau \), with \( \tau \in \mathbb{H}^2 \). Then we consider the corresponding absolute sum

\[
\sum_{\lambda \in \mathbb{Z} + \mathbb{Z}\tau, \lambda \neq 0} \frac{1}{|\lambda|^k} = \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{|m + n\tau|^k}.
\]

(29.3.4)

The number of pairs \((m, n)\) with \( r \leq |m\tau + n| < r + 1\) is the number of lattice points in an annulus of area \( \pi (r + 1)^2 - \pi r^2 = O(r) \), so there are \( O(r) \) such points; and thus the series (29.3.4) is majorized by (a constant multiple of) \( \sum_{r=1}^{\infty} r^{1-k} \), which is convergent for \( k > 2 \). \( \square \)

29.3.5. For \( z \in \mathbb{H}^2 \) and \( k \in 2\mathbb{Z}_{>2} \), define

\[
G_k(z) = G_k(\mathbb{Z} + \mathbb{Z}z) = \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(m + n\tau)^k}.
\]

Lemma 29.3.6. \( G_k(z) \) is holomorphic for \( z \in \mathbb{H}^2 \), and

\[
G_k(\gamma z) = (cz + d)^k G_k(z)
\]

for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) \).

Proof. This is true for \( z \in \mathbb{H} \) since then

\[
|m + nz|^2 = m^2 + 2mn \text{Re } z + n^2|z|^2 \geq m^2 - mn + n^2 = |m + n\omega|^2
\]

thus \( |G_k(z)| \leq |G_k(\omega)| \) and so by the Weierstrass \( M \)-test, \( G_k(z) \) is holomorphic for \( z \in \mathbb{H} \) (by Morera’s theorem, uniformly convergence implies holomorphicity). But now for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), we claim that

\[
G_k(\gamma z) = (cz + d)^k G_k(z)
\]

(and note this does not depend on the choice of sign): indeed, we have

\[
\frac{1}{m(gz) + n} = \frac{cz + d}{(am + cn)z + (bm + dn)}
\]
and the map

\[(m, n) \mapsto (m, n) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (am + cn, bm + dn)\]

is a permutation of \(\mathbb{Z}^2 - \{(0, 0)\}\), so by absolute convergence we may rearrange the sum and we have

\[G_k(\gamma z) = (cz + d)^k G_k(z);\]

by transport, since \(\Gamma \mathbb{H} = \mathbb{H}^2\), we see that \(G_k(z)\) is holomorphic on all of \(\mathbb{H}^2\).

\[29.3.7.\] To produce holomorphic functions that are well-defined on the quotient \(\Gamma \mathbb{H}^2\), we can take ratios of Eisenstein series; in the next section, we exhibit a map

\[j : \mathbb{H}^2 \to \mathbb{C}\]

obtained in this way that defines a biholomorphic identification \(\Gamma \mathbb{H}^2 \to \mathbb{C}\) (Theorem 29.4.13).

\[29.4\] Classical modular forms

In the previous section, we saw that natural sums (Eisenstein series) defined functions on \(\mathbb{H}^2\) that transformed with respect to \(\Gamma = \text{PSL}_2(\mathbb{Z})\) with a natural invariance. In this section, we pursue this more systematically.

**Definition 29.4.1.** Let \(k \in 2\mathbb{Z}\). A map \(f : \mathbb{H}^2 \to \mathbb{C}\) is \(\text{weight } k\) \(\text{invariant}\) under \(\Gamma\) if

\[f(\gamma z) = (cz + d)^k f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}).\]  

(29.4.2)

**29.4.3.** Since \(\text{PSL}_2(\mathbb{Z})\) is generated by \(S, T\), a map \(f\) is \(\text{weight } k\) \(\text{invariant}\) if and only if both

\[f(z + 1) = f(z)\]

\[f(-1/z) = z^k f(z)\]

for all \(z \in \mathbb{H}^2\). Put a third way, since

\[\frac{d(\gamma z)}{dz} = \frac{1}{(cz + d)^2}\]

the weight \(k\) invariance (29.4.2) can be rewritten

\[f(\gamma z) d(\gamma z)^{\otimes k} = f(z) d z^{\otimes k}\]

so the expression \(f(z) d z^{\otimes k}\) is invariant under \(\Gamma\).
CHAPTER 29. CLASSICAL MODULAR GROUP

Let \( f : H^2 \to \mathbb{C} \) be a meromorphic map that is weight \( k \) invariant under \( \Gamma \). Then \( f(z+1) = f(z) \), and if \( f \) admits a Fourier series expansion in \( q = \exp(2\pi i z) \)

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n q^n \in \mathbb{C}((q))
\]

with \( a_n \in \mathbb{C} \), then we say that \( f \) is meromorphic at \( \infty \), and if further \( a_n = 0 \) for \( n < 0 \), we say \( f \) is holomorphic at \( \infty \).

Definition 29.4.4. Let \( k \in 2\mathbb{Z} \). A meromorphic modular form of weight \( k \) is a meromorphic map \( f : H^2 \to \mathbb{C} \) that is weight \( k \) invariant under \( \Gamma \) and meromorphic at \( \infty \). A meromorphic modular function is a meromorphic modular form of weight \( 0 \).

A (holomorphic) modular form of weight \( k \) is a holomorphic map \( f : H^2 \to \mathbb{C} \) that is weight \( k \) invariant under \( \Gamma \) and holomorphic at \( \infty \).

Lemma 29.4.5. The Eisenstein series \( G_k(z) \) is a holomorphic modular form of weight \( k \in 2\mathbb{Z}_{\geq 2} \) with Fourier expansion

\[
G_k(z) = 2\zeta(k) + 2 \left( \frac{2\pi i}{k-1} \right)^k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n
\]

where

\[
\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}
\]

and

\[
\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.
\]

Proof. We start with the formula

\[
\pi \cot(\pi z) = \sum_{m=-\infty}^{\infty} \frac{1}{z + m} = \lim_{M \to \infty} \sum_{m=-M}^{M} \frac{1}{z + m}
\]

(Exercise 29.10), since

\[
\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = i \left( q + 1 \right) \left( q - 1 \right) = i \left( q + 1 \right) \left( q - 1 \right)
\]

we obtain the Fourier expansion

\[
\pi \cot(\pi z) = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n.
\]

Equating (29.4.6)-(29.4.7) and differentiating \( k - 1 \) times, we find that

\[
\sum_{m=0}^{\infty} \frac{1}{(m+z)^k} = \frac{1}{(k-1)!} (2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n
\]
(since $k$ is even). Thus

$$G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m+nz)^k} = 2\zeta(k) + 2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m+nz)^k}$$

so replacing $n \leftarrow a$ in (29.4.8) and then substituting $z \leftarrow nz$, summing we obtain

$$G_k(z) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{d=1}^{\infty} \sum_{a=1}^{\infty} d^{k-1} q^a d^n$$

$$= 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \left( \sum_{d|n} d^{k-1} \right) q^n$$

as claimed. The fact that $G_k$ is holomorphic at $\infty$ then follows by definition.

We accordingly define the normalized Eisenstein series by

$$E_k(z) = \frac{1}{2\zeta(k)} G_k(z) = 1 + \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

(Exercise 29.11), and $B_k \in \mathbb{Q}^\times$ are Bernoulli numbers. We have

$$E_4(z) = 1 + 240q + 2160q^2 + \ldots$$
$$E_6(z) = 1 - 504q - 16632q^2 + \ldots$$

29.4.9. A meromorphic modular function descends to a function on $\Gamma \backslash \mathbb{H}^2 = Y$; although this is not true for forms of weight $k \neq 0$, the order of pole or zero $\text{ord}_z(f)$ is well defined on the orbit $\Gamma z$. The order of the stabilizer $e_z = \# \text{Stab}_\Gamma(z)$ is well defined on the orbit $\Gamma z$, since points in the same orbit have conjugate stabilizers.

If $f$ is a meromorphic modular form, then $f$ has only a finite number of zeros or poles in $Y$ (i.e., only finitely many $\Gamma$-orbits of zeros or poles): indeed, since $f$ is meromorphic at $\infty$, there exists $\epsilon > 0$ such that $f$ has no zero or pole with $0 < |q| < \epsilon$, i.e., with $\text{Im } z > M = \log(1/\epsilon)/(2\pi)$; but the part of $\mathfrak{F}$ with $\text{Im } z \leq M$ is compact, and since $f$ is meromorphic in $\mathbb{H}^2$, it has only a finite number of zeros or poles.

Proposition 29.4.10. Let $f : \mathbb{H}^2 \to \mathbb{C}$ be a meromorphic modular function of weight $k$, not identically zero. Then

$$\text{ord}_\infty(f) + \sum_{\Gamma z \in \Gamma \backslash \mathbb{H}^2} \frac{1}{e_z} \text{ord}_z(f) = \frac{k}{12}. \quad (29.4.11)$$

The sum in Proposition 29.4.11 makes sense, by Paragraph 29.4.9
Proof. To prove this theorem, we perform a contour integration \( \frac{1}{2\pi i} \oint_C \frac{df}{f} \) on the boundary of \( \mathbb{H} \). More precisely, first assume that \( f \) has neither zero nor pole on the boundary of \( \mathbb{H} \) except possibly at \( i, \omega, -\omega^2 \). We consider the contour \( C \) containing all zeros or poles of \( f \) in \( \text{int}(\mathbb{H}) \):

![Contour Diagram]

By the argument principle, we have

\[
\frac{1}{2\pi i} \oint_C \frac{df}{f} = \sum_{z \in \text{int}(\mathbb{H})} \text{ord}_z(f).
\]

We write \( C \) as the sum of several contours as indicated. In the change of variable \( z \to q = e^{2\pi iz} \), the contour \( C_1 \) transforms into a circle centered at \( q = 0 \) with negative orientation whose only enclosed zero or pole is \( \infty \). Thus

\[
\frac{1}{2\pi i} \int_{C_1} = -\text{ord}_\infty(f).
\]

We have \( T^{-1}(C_8) = C_2 \) with opposite orientation; since \( f(z + 1) = f(z) \), these contributions cancel. On \( C_3 \), as the radius of this arc of a circle tends to 0, we have

\[
\frac{1}{2\pi i} \int_{C_2} \frac{df}{f} \to \frac{1}{2\pi i} \left( \frac{-\pi i}{3} \right) = \frac{1}{6}
\]

as the angle formed with \( \omega \) by the endpoints of \( C_2 \) is \( \pi/3 \) (Exercise 29.8). Similarly,

\[
\frac{1}{2\pi i} \int_{C_5} \frac{df}{f} \to -\frac{1}{2} \quad \text{and} \quad \frac{1}{2\pi i} \int_{C_7} \frac{df}{f} \to -\frac{1}{6}.
\]
Finally, we have \( S(C_6) = C_4 \) with opposite orientation; but now \( f(Sz) = z^k f(z) \) so
\[
\frac{df(Sz)}{dz} = k z^{k-1} f(z) + z^k \frac{df(z)}{dz}
\]
and hence
\[
\frac{df(Sz)}{f(Sz)} = k \frac{dz}{z} + \frac{df(z)}{f(z)}
\]
and thus taking the difference we obtain
\[
\frac{1}{2\pi i} \int_{C_4 \cup C_6} \frac{df}{f} = \frac{1}{2\pi i} \int_{C_4} -k \frac{dz}{z} \rightarrow -k \frac{-\pi i}{2\pi i} \left( \frac{-\pi i}{6} \right) = \frac{k}{12}
\]
as the angle formed with 0 is now \( \pi/6 \). Summing, we obtain the result.

If \( f \) has a zero or pole on the boundary of \( \mathbb{H} \), we repeat the same argument with a contour modified as follows:

The details are requested in Exercise 29.9.

**29.4.12.** We have \( E_4(STz) = (z+1)^4 E_4(z) \), so since \( (ST)(\omega) = \omega \), we have
\[
E_4(\omega) = (\omega + 1)^4 E_4(\omega) = \omega^2 G_4(\omega)
\]
so \( E_4(\omega) = 0 \). By Proposition 29.4.10 we conclude that \( E_4(z) \) has no other zeros (or poles) in \( \mathbb{H} \). Similarly,
\[
E_6(i) = E_6(Si) = i^6 E_6(i) = -E_6(i)
\]
so \( E_6(i) = 0 \) and \( E_6(z) \) has no other zeros.

For the same reason, the function
\[
\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = q - 24q^2 + 252q^3 - \ldots
\]
is a modular form of weight 12 with no zeros in \( \mathbb{H}^2 \).
Theorem 29.4.13. The function
\[ j(z) = \frac{E_4(z)^3}{\Delta(z)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 86429970q^3 + \ldots \]
is a meromorphic modular function, holomorphic in \( \mathbf{H}^2 \), defining a biholomorphic identification
\[ Y = \Gamma \backslash \mathbf{H}^2 \rightarrow \mathbb{C}. \]

Proof. The function \( j \) is weight 0 invariant under \( \Gamma \) as the ratio of two forms that are weight 12 invariant. Since \( E_4 \) is holomorphic in \( \mathbf{H}^2 \), and \( \Delta \) is holomorphic and has no zeros in \( \mathbf{H}^2 \), the ratio is holomorphic. We have that \( j(z) \) has a simple pole at \( z = \infty \), corresponding to a simple zero of \( \Delta \) at \( z = \infty \). We have \( j(i) = 1728 \), and \( j(z) - 1728 \) has a double zero at \( i \), and \( j(\omega) = 0 \) is a triple zero.

To conclude that \( j \) defines a biholomorphic identification, we show that \( j(z) - c \) has a unique zero \( \Gamma z \in Y \). If \( c \neq 0, 1728 \), this follows immediately from Proposition 29.4.10; if \( c = 0, 1728 \), the results follow for the same reason from the multiplicity of the zero. \( \square \)

29.5 Congruence subgroups

The finite-index subgroups of \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) play a central role in what follows, and of particular importance are those subgroups defined by congruence conditions on the entries.

Definition 29.5.1. Let \( N \in \mathbb{Z}_{\geq 1} \). The congruence subgroup \( \Gamma(N) \subseteq \Gamma \) of level \( N \) is
\[ \Gamma(N) = \ker(\text{PSL}_2(\mathbb{Z}) \rightarrow \text{PSL}_2(\mathbb{Z}/N\mathbb{Z})) \]
\[ = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) : a \equiv d \equiv \pm 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\} \]
\[ = \left\{ \gamma \in \text{PSL}_2(\mathbb{Z}) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}. \]

To avoid confusion, we will now write \( \Gamma(1) = \text{PSL}_2(\mathbb{Z}) \).

29.5.2. By strong approximation for \( \text{SL}_2(\mathbb{Z}) \) (Theorem 24.1.1), the map \( \text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) is surjective for all \( N \geq 1 \), so we have an exact sequence
\[ 1 \rightarrow \Gamma(N) \rightarrow \Gamma(1) \rightarrow \text{PSL}_2(\mathbb{Z}/N\mathbb{Z}) \rightarrow 1. \]

Definition 29.5.3. A subgroup \( \Gamma \leq \Gamma(1) \) is a congruence subgroup if \( \Gamma \geq \Gamma(N) \) for some \( N \geq 1 \).
29.5.4. In addition to the congruence groups \( \Gamma(N) \) themselves, we will make use of two other important congruence subgroups for \( N \geq 1 \):

\[
\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}
\]

\[
= \left\{ \gamma \in \text{PSL}_2(\mathbb{Z}) : \gamma \equiv \pm \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\} \tag{29.5.5}
\]

\[
\Gamma_1(N) = \left\{ \gamma \in \text{PSL}_2(\mathbb{Z}) : \gamma \equiv \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}
\]

Visibly, we have \( \Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N) \).

In the remainder of this section, we consider as an extended example the case \( N = 2 \). We can equally well write

\[
\Gamma(2) = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) : b \equiv c \equiv 0 \pmod{2} \right\}
\]

We have

\[
\Gamma(1)/\Gamma(2) \simeq \text{PSL}_2(\mathbb{Z}/2\mathbb{Z}) = \text{GL}_2(\mathbb{F}_2) \simeq S_3
\]

the nonabelian group of order 6, so in particular \( [\Gamma(1) : \Gamma(2)] = 6 \).

We can uncover the structure of the group \( \Gamma(2) \) in a manner similar to what we did for \( \Gamma(1) \) in section 29.1—the details are requested in Exercise 29.12. The group \( \Gamma(2) \) is generated by

\[
T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}
\]

which act on \( \mathcal{H} \) by \( z \mapsto z + 2 \) and \( z \mapsto z/(2z + 1) \), respectively, and a fundamental set \( \Box \) is given by

![Diagram](image-url)
(In fact, later we will see from more general structural results on fundamental domains that \( \Gamma(2) \) is freely generated by these two elements, so it is isomorphic to the free group on two generators.)

The action \( \Gamma(2) \act \mathbb{H}^2 \) is free: by Paragraph 29.1.7, if \( \gamma z = z \) with \( \gamma \in \Gamma(2) \leq \Gamma(1) \), then \( \gamma \) is conjugate in \( \Gamma(1) \) to either \( S, ST \); but \( \Gamma(2) \triangleleft \Gamma(1) \) is normal, so this implies that either \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) or \( ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \) belongs to \( \Gamma(2) \), a contradiction.

Let \( Y(2) = \Gamma(2) \backslash \mathbb{H}^2 \). Then gluing together the fundamental set, we have a homeomorphism

\[
Y(2) \simeq \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}
\]

Indeed, the limit points of \( \mathbb{H} \) in \( \partial \mathbb{H}^2 \) are the points \(-1, 0, 1, \infty\) and the points \(-1, 1\) are identified in the quotient (by translation). The orbit of these points under \( \Gamma(2) \) is \( \mathbb{P}^1(\mathbb{Q}) \subseteq \partial \mathbb{H}^2 \), so letting \( \mathcal{H}^* = \mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q}) \), we have a homeomorphism

\[
X(2) = \Gamma(2) \backslash \mathcal{H}^* \simeq \mathbb{P}^1(\mathbb{C})
\]

As with \( X(1) \), this homeomorphism can be given by a holomorphic map

\[
\lambda : X(2) \sim \mathbb{P}^1(\mathbb{C})
\]

the map \( \lambda \) is invariant under \( z \mapsto z + 2 \), so has a Fourier expansion in terms of \( q^{1/2} = e^{\pi i z} \), and explicitly we have

\[
\lambda(z) = 16q^{1/2} - 128q^{3/2} - 3072q^2 + 11488q^{5/2} - 38400q^3 + \ldots \tag{29.5.6}
\]

and \( \lambda(\gamma z) = \lambda(z) \) for all \( \gamma \in \Gamma(2) \).

As a uniformizer for a congruence subgroup of \( \text{PSL}_2(\mathbb{Z}) \), the function \( \lambda(z) \) has a modular interpretation: there is a family of elliptic curves over \( X(2) \) equipped with extra structure. Specifically, given \( \lambda \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \), the corresponding elliptic curve with extra structure is given by the Legendre curve

\[
E_\lambda : y^2 = x(x - 1)(x - \lambda),
\]
29.6. EXTENSIONS AND FURTHER READING

equipped with the isomorphism \((\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{\sim} E[2]\) determined by sending the standard generators to the 2-torsion points \((0, 0)\) and \((1, 0)\). There is a forgetful map that forgets this additional torsion structure on a Legendre curve and remembers only isomorphism class; on the algebraic level, this corresponds to an expression of \(j\) in terms of \(\lambda\), which is given by

\[
j(\lambda) = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2};
\]

(29.5.7)

From (29.5.7) (and the first term), the complete Puiseux series expansion (29.5.6) can be obtained recursively.

The congruence conditions (29.5.5) imply that \(\Gamma_0(2) = \Gamma_1(2)\) has index 2 in \(\Gamma(2)\), with the quotient generated by \(T\), and we obtain a fundamental set by identifying the two ideal triangles in \(\Pi\) above.

29.6 Extensions and further reading

Exercises

29.1. Prove that \(Y(1) = \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}^2\) has area\((Y(1)) = \pi/3\) by direct integration (verifying the Gauss–Bonnet formula).

29.2. Show that \(\text{PSL}_2(\mathbb{Z})\) is generated by \(T\) and \(U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\).

29.3. In this exercise, we link the fact that \(\text{PSL}_2(\mathbb{Z})\) is generated by \(S, T\) to a kind of continued fraction via the Euclidean algorithm. Let \(a, b \in \mathbb{Z}_{\geq 1}\) with \(a \geq b\).

a) Show that there exist unique \(q, r \in \mathbb{Z}\) such that \(a = qb - r\) and \(q \geq 2\) and \(0 \leq r < b\).

From (a), define inductively \(r_0 = a, r_1 = b, \text{ and } r_{i-1} = qr_i - r_{i+1}\) with \(0 \leq r_{i+1} < r_i\); we then have \(r_1 > r_2 > \cdots > r_t > r_{t+1} = 0\) for some \(t > 0\).

b) Show that gcd\((a, b) = r_t, \text{ and if gcd}(a, b) = 1\) then

\[
\frac{a}{b} = q_1 - \frac{1}{q_2 - \frac{1}{\cdots - \frac{1}{q_t}}}.
\]

This kind of continued fraction is called a negative-regular or Hirzebruch–Jung continued fraction \([\text{Jun08, Hir53}]\).

c) Show (by induction) that

\[
\begin{pmatrix} q_t & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ r_t \end{pmatrix}.
\]

Show that

\[
\begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix} \in \langle S, T \rangle
\]
and interpret the action of a matrix \( \begin{pmatrix} q & -1 \\ 1 & 0 \end{pmatrix} \) with \( q \in \mathbb{Z} \) in terms of the reduction algorithm to the fundamental set \( \mathfrak{p} \) for \( \text{PSL}_2(\mathbb{Z}) \).

d) Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \). Show that there exists \( W \in \langle S, T \rangle \) such that
\[
AW = \begin{pmatrix} 1 & 0 \\ c' & 1 \end{pmatrix}
\]
with \( c' \in \mathbb{Z} \), and conclude that \( \langle S, T \rangle = \text{PSL}_2(\mathbb{Z}) \).

29.4. In the proof of Lemma 29.1.1 we showed that the reduction algorithm terminates after finitely many steps directly. Prove this again using the fact that \( \Gamma \) is discrete and so Theorem 28.3.12(ii) holds.

29.5. Compute the class number \( h(d) \) and the set of reduced (positive definite) binary quadratic forms of discriminant \( d = -71 \).

29.6. Let \( Q_d \) be the set of primitive, positive definite binary quadratic forms of discriminant \( d < 0 \).

a) Show that the group \( \text{GL}_2(\mathbb{Z}) \) acts on \( Q_d \), with \( \text{PGL}_2(\mathbb{Z}) \) acting faithfully.

b) Consider the action of \( \text{PGL}_2(\mathbb{Z}) \) on \( H^2 \). Show that every \( Q \in Q_d \) is equivalent to a \( \text{GL}_2(\mathbb{Z}) \)-reduced form \( ax^2 + bxy + cy^2 \) satisfying
\[
0 \leq b \leq a \leq c.
\]

[Hint: Find a nice fundamental set \( \mathfrak{p} \) for \( \text{PGL}_2(\mathbb{Z}) \).]

c) By transport, Paragraph 29.1.7 computes the stabilizer of \( \text{PSL}_2(\mathbb{Z}) \) on \( Q_d \). Compute \( \text{Stab}_{\text{PGL}_2(\mathbb{Z})}(Q) \) for \( Q \in Q_d \).

29.7. Let
\[
S = \mathbb{Z} \oplus \mathbb{Z}[(d + \sqrt{d})/2] \subset K = \mathbb{Q}(\sqrt{d})
\]
be the quadratic ring of discriminant \( d < 0 \). Let \( \text{Pic}(S) \) be the group of invertible fractional ideals of \( S \) modulo principal ideals. Show that the map
\[
Q_d/\Gamma \rightarrow \text{Pic}(S)
\]
\[
[ax^2 + bxy + cy^2] \mapsto [a] = \left[ \begin{pmatrix} a, & -b + \sqrt{d} \\ 2 \end{pmatrix} \right]
\]
is a bijection, where \( Q_d/\Gamma \) is the set of (\( \text{SL}_2(\mathbb{Z}) \))-equivalence classes of (primitive, positive definite) binary quadratic forms of discriminant \( d \).

29.8. Let \( f : U \rightarrow \mathbb{C} \) be a meromorphic function in an open neighborhood \( U \supset \mathbb{C} \) with \( 0 \in U \), and let \( C \) be the contour along the arc of a circle of radius \( \epsilon > 0 \) with total angle \( \theta \). Show that
\[
\lim_{\epsilon \to 0} \int_C \frac{df}{f} = \theta i.
\]
29.6. EXTENSIONS AND FURTHER READING

29.9. Complete the proof of Proposition 29.4.10 in case \( f \) has poles or zeros along the boundary of \( \Xi \).

29.10. Prove the formula

\[
\pi \cot(\pi z) = \sum_{m=-\infty}^{\infty} \frac{1}{z + m}
\]

for \( z \in \mathbb{C} \). [Hint: the difference \( h(z) \) of the left- and right-hand sides is bounded away from \( \mathbb{Z} \), invariant under \( z \mapsto T(z) = z + 1 \), and in a neighborhood of 0 is holomorphic (both sides have principal part \( 1/z \) so bounded, so \( h(z) \) is bounded in \( \mathbb{C} \) and hence constant.]

29.11. In this exercise, we give Euler’s evaluation of \( \zeta(k) \) in terms of Bernoulli numbers. Define the series

\[
\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} = 1 - \frac{x}{2} + \frac{x^2}{6!} - \frac{x^4}{30!} \ldots \in \mathbb{Q}[[x]]. \tag{29.6.1}
\]

a) Plug in \( x = 2iz \) into (29.6.1) to obtain

\[
z \cot z = 1 + \sum_{k=2}^{\infty} \frac{B_k (2iz)^k}{k!}.
\]

b) Take the logarithmic derivative of

\[
\sin z = z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2 \pi^2} \right)
\]

to show

\[
z \cot z = 1 - 2 \sum_{k=2}^{\infty} \sum_{k=1}^{\infty} \left( \frac{z}{n\pi} \right)^k.
\]

c) Conclude that

\[
\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = -\frac{1}{2} \frac{(2\pi i)^k}{k!} B_k
\]

for \( k \in 2\mathbb{Z}_{\geq 1} \).

29.12. Show that the elements

\[
\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}
\]

generate \( \Gamma(2) \) in a similar manner as section 29.1 using the fundamental set

\( \Xi = \{ z \in \mathbb{H}^2 : |\text{Re} \ z| \leq 1, |2z \pm 1| \geq 1 \} \).

29.13. [[Exercise with \( \Gamma_0(2) = \Gamma_1(2) \)]]