Chapter 31

Fundamental domains

We have seen in sections 29.1 and 30.5 that understanding a nice fundamental set for the action of a discrete group Γ is not only useful to visualize the action of the group by selecting representatives of the orbits, but it is also instrumental for many other purposes—including understanding the structure of the group itself. In this chapter, we pursue a general construction of nice fundamental domains for the action of a discrete group of isometries.

31.1 Dirichlet domains for Fuchsian groups

In this introductory section, we preview the results in this chapter specialized to the case of Fuchsian groups. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group; then Γ is discrete, acting properly by isometries on the hyperbolic plane H^2, with metric $\rho(\cdot,\cdot)$ and hyperbolic area μ.

A natural way to produce fundamental sets that provide appealing tesselations of H^2 is to select in each orbit the points closest to a fixed point $z_0 \in \mathsf{H}^2$, as follows.

Definition 31.1.1. The *Dirichlet domain* for Γ centered at $z_0 \in \mathsf{H}^2$ is

$$\text{¤}(\Gamma; z_0) = \{z \in \mathsf{H}^2 : \rho(z, z_0) \leq \rho(\gamma z, z_0) \text{ for all } \gamma \in \Gamma\}.$$

As the group Γ will not vary, we suppress the dependence on Γ and often write simply $\text{¤}(z_0) = \text{¤}(\Gamma; z_0)$.

31.1.2. The set $\text{¤}(z_0)$ is an intersection

$$\text{¤}(z_0) = \bigcap_{\gamma \in \Gamma} H(\gamma; z_0) \tag{31.1.3}$$

where

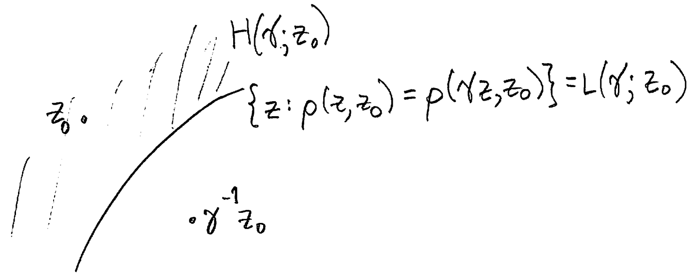
$$H(\gamma; z_0) = \{z \in \mathsf{H}^2 : \rho(z, z_0) \leq \rho(\gamma z, z_0) = \rho(z, \gamma^{-1} z_0)\}. \tag{31.1.4}$$

In particular, since each $H(\gamma; z_0)$ is closed, we conclude from (31.4.12) that $\text{¤}(z_0)$ is closed.

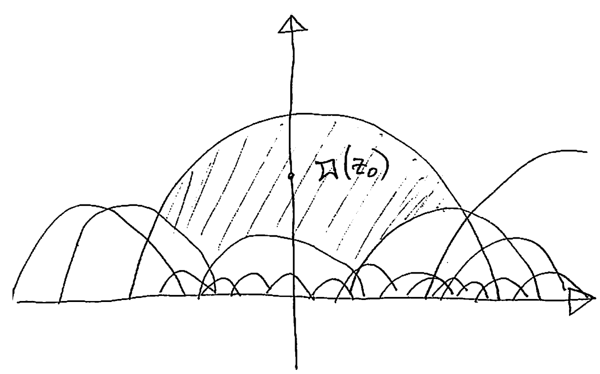
The sets $H(\gamma; z_0)$ can be further described as follows. If $z_0 = \gamma^{-1}z_0$, then $H(\gamma; z_0) = \mathbf{H}^2$. So suppose $z_0 \neq \gamma^{-1}z_0$. Then by Exercise 27.8, $H(\gamma; z_0)$ is a (half!) half-plane consisting of the set of points as close to z_0 as $\gamma^{-1}z_0$, and $H(\gamma; z_0)$ is convex (if two points lie in the half-plane then so does the geodesic between them). The boundary

$$\operatorname{bd} H(\gamma; z_0) = L(\gamma; z_0) = \{z \in \mathbf{H}^2 : \rho(z, z_0) = \rho(z, \gamma^{-1}z_0)\}$$

is the perpendicular bisector of the geodesic segment from z_0 to $\gamma^{-1}z_0$, and $L(\gamma; z_0)$ is geodesic.



From the description in Paragraph 31.1.2, the sketch of a Dirichlet domain looks like:



Dirichlet domains are ubiquitous, and already the fundamental sets we have seen are in fact examples of Dirichlet domains.

Example 31.1.5. We claim that the Dirichlet domain for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ centered at $z_0 = 2i$ is in fact the fundamental set for Γ introduced in section 29.1, i.e.,

$$⌑(2i) = \{z \in \mathbf{H}^2 : |\operatorname{Re} z| \leq 1/2,\ |z| \geq 1\}. \tag{31.1.6}$$

Recall the generators $S, T \in \Gamma$ with $Sz = -1/z$ and $Tz = z + 1$. By (27.4.3) we have

$$\cosh \rho(z, 2i) = 1 + \frac{|z - 2i|^2}{4 \operatorname{Im} z}.$$

Let $z \in \mathbf{H}^2$. Visibly, we have

$$\rho(z, 2i) \le \rho(Tz, 2i) \quad \Leftrightarrow \quad \operatorname{Re} z \ge -1/2 \tag{31.1.7}$$

or put another way

$$H(T; 2i) = \{z \in \mathbf{H}^2 : \operatorname{Re} z \ge -1/2\}.$$

Similarly, we have $H(T^{-1}; 2i) = \{z \in \mathbf{H}^2 : \operatorname{Re} z \le 1/2\}$. Equivalently, the geodesic perpendicular bisector between $2i$ and $2i \pm 1$ are the lines $\operatorname{Re} z = \pm 1/2$.

In the same manner, we find that

$$\begin{aligned} \rho(z, 2i) \le \rho(Sz, 2i) \quad &\Leftrightarrow \quad \frac{|z - 2i|^2}{\operatorname{Im} z} \le \frac{|(-1/z) - 2i|^2}{\operatorname{Im}(-1/z)} \\ &\Leftrightarrow \quad \frac{|z - 2i|^2}{\operatorname{Im} z} \le \frac{4|z|^2 |z - i/2|^2}{|z|^2 \operatorname{Im} z} \\ &\Leftrightarrow \quad |z - 2i|^2 \le 4|z - i/2|^2 \\ &\Leftrightarrow \quad |z| \ge 1 \end{aligned} \tag{31.1.8}$$

so $H(S; 2i) = \{z \in \mathbf{H}^2 : |z| \ge 1\}$. To prove this another way, note that the geodesic between $2i$ and $S(2i) = (1/2)i$ is along the imaginary axis with midpoint at i, and so the perpendicular bisector $L(S; 2i)$ is the unit semicircle.

The containment $(\subseteq)$ in (31.1.6) then follows directly from (31.1.7)–(31.1.8). Conversely, we show the containment $(\supseteq)$ for the interior—since $\mathcal{D}(2i)$ is closed, this implies the full containment. Let $z \in \mathbf{H}^2$ have $|\operatorname{Re} z| < 1/2$ and $|z| > 1$, and suppose that $z \notin \mathcal{D}(2i)$; then there exists $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ such that $z' = \gamma z$ has $\rho(z', 2i) < \rho(z, 2i)$, without loss of generality (replacing z' by Sz' or Tz') we may assume $|\operatorname{Re} z'| \le 1/2$ and $|z'| \ge 1$; but then by Lemma 29.1.3, we conclude that $z' = z$, a contradiction.

(Note that the same argument works with $z_0 = ti$ for any $t \in \mathbb{R}_{>1}$.)

With this example in hand, we see that Dirichlet domains have quite nice structure. To make this more precise, we upgrade our notion of fundamental set (Definition 28.1.13) as follows.

Definition 31.1.9. A fundamental set $\mathcal{D}$ for Γ is *locally finite* if for each compact set $K \subset \mathbf{H}^2$, we have $\gamma K \cap \mathcal{D} \ne \emptyset$ for only finitely many $\gamma \in \Gamma$.

A *fundamental domain* for $\Gamma \circlearrowright \mathbf{H}^2$ is a fundamental set $\mathcal{D} \subseteq \mathbf{H}^2$ such that $\mu(\operatorname{bd} \mathcal{D}) = 0$.

The first main result of this section is as follows (Theorem 31.5.3). We must first find a point z_0 such that no nonidentity element of Γ fixes z_0:

Theorem 31.1.10. *Let $z_0 \in \mathbf{H}^2$ satisfy $\operatorname{Stab}_\Gamma(z_0) = \{1\}$. Then $\mathcal{D}(z_0)$ is a connected, convex, locally finite fundamental domain for Γ with geodesic boundary.*

Specifically, in any bounded set A, there are finitely many $\gamma_i \in \Gamma - \{1\}$ such that

$$A \cap \operatorname{bd} \mathcal{D}(z_0) \subseteq \bigcup_i L(z_0, \gamma_i^{-1} z_0).$$

31.2 Ford domains

In this section, we reinterpret Dirichlet domains in the unit disc $\mathbf{D}^2$, as it is more convenient to compute and visualize distances this model. Let $z_0 \in \mathbf{H}^2$. We apply the the map (27.6.3)

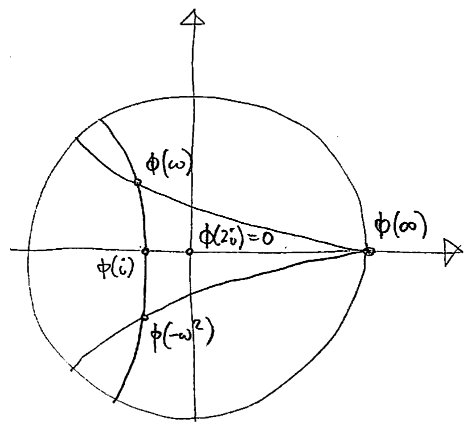
$$\phi : \mathbf{H}^2 \to \mathbf{D}^2$$
$$w = \frac{z - z_0}{z - \overline{z_0}}$$

with $z_0 \mapsto \phi(z_0) = w_0 = 0$. Then by (27.6.6) we have that

$$\rho(w, 0) = \log \frac{1 + |w|}{1 - |w|} = 2 \tanh^{-1} |w| \tag{31.2.1}$$

is an increasing function of $|w|$.

Example 31.2.2. The Dirichlet domain from Example 31.1.5 looks like the following in $\mathbf{D}^2$ (with $z_0 = 2i$):



Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group, and (recalling Paragraph 27.6.7) to ease notation, we identify Γ with Γ^ϕ. We analogously define a Dirichlet domain $\text{⌑}(w_0)$ for a Fuchsian group Γ centered at $w_0 \in \mathbf{D}^2$ and we have

$$\phi(\text{⌑}(z_0)) = \text{⌑}(w_0) \subset \mathbf{D}^2$$

if $\phi(z_0) = w_0$. In particular, the statement of Theorem 31.1.10 applies equally well to $\text{⌑}(w_0) \subseteq \mathbf{D}^2$.

For simplicity (and without loss of generality), we only consider the case where $w_0 = 0$, and then from (31.2.1) we have

$$\text{⌑}(\Gamma; 0) = \{w \in \mathbf{D}^2 : |w| \leq |\gamma w| \text{ for all } \gamma \in \Gamma\}. \tag{31.2.3}$$

31.2.4. We now pursue a tidy description of the set (31.2.3). Let

$$g = \begin{pmatrix} \overline{d} & \overline{c} \\ c & d \end{pmatrix} \in \mathrm{PSU}(1,1) \circlearrowright \mathbf{D}^2$$

with $c, d \in \mathbb{C}$ satisfying $|d|^2 - |c|^2 = 1$. From (31.2.1), we have $\rho(w,0) \le \rho(gw,0)$ if and only if

$$|w| \le \left| \frac{\overline{d}w + \overline{c}}{cw + d} \right|; \tag{31.2.5}$$

expanding out (31.2.5) and with a bit of patience (Exercise 31.5), we see that this is equivalent to simply

$$|cw + d| \ge 1.$$

But we can derive this more easily using the invariance of the metric: the hyperbolic metric (Definition 27.6.1) on $\mathbf{D}^2$ is invariant $\mathrm{d}(gs) = \mathrm{d}s$, so

$$\mathrm{d}s = \frac{|\mathrm{d}w|}{(1 - |w|)^2} = \frac{|\mathrm{d}(gw)|}{(1 - |gw|)^2} = \mathrm{d}(gs)$$

so by the chain rule

$$\left| \frac{\mathrm{d}g}{\mathrm{d}w} \right| = \left(\frac{1 - |gw|}{1 - |w|} \right)^2.$$

Thus

$$|w| \le |gw| \quad \Leftrightarrow \quad \left| \frac{\mathrm{d}g}{\mathrm{d}w} \right| \le 1; \tag{31.2.6}$$

but then we compute

$$\frac{\mathrm{d}g}{\mathrm{d}w} = \frac{(cw + d)\overline{d} - (\overline{d}w + \overline{c})c}{(cw + d)^2} = \frac{1}{(cw + d)^2} \tag{31.2.7}$$

so $|w| \le |gw|$ if and only if $|cw + d| \ge 1$.

The equivalence (31.2.6) shows that $\rho(w,0) = \rho(gw,0)$ if and only if $\left| \frac{\mathrm{d}g}{\mathrm{d}w} \right| = 1$ if and only if g acts as a *Euclidean* isometry at the point w (preserving the length of tangent vectors in the Euclidean metric). So we are led to make the following definition.

Definition 31.2.8. Let $g = \begin{pmatrix} \overline{d} & \overline{c} \\ c & d \end{pmatrix} \in \mathrm{PSU}(1,1)$. The *isometric circle* of g is

$$I(g) = \left\{ w \in \mathbb{C} : \left| \frac{\mathrm{d}g}{\mathrm{d}w} \right| = 1 \right\} = \{ w \in \mathbb{C} : |cw + d| = 1 \}.$$

31.2.9. If $c \ne 0$, then $I(g)$ is a circle with radius $1/|c|$ and center $-d/c \in \mathbb{C}$, and if $c = 0$ then $gw = (\overline{d}/d)w$ with $|\overline{d}/d| = 1$ is rotation about the origin. The condition $c \ne 0$ is equivalent to $g(0) \ne 0$, i.e., $g \notin \mathrm{Stab}_{\mathrm{PSU}(1,1)}(0)$.

We define interior and exterior of $I(g)$ by

$$\begin{aligned}\operatorname{int} I(g) &= \{w \in \mathbb{C} : |cw+d| < 1\}\\ \operatorname{ext} I(g) &= \{w \in \mathbb{C} : |cw+d| > 1\}.\end{aligned}$$

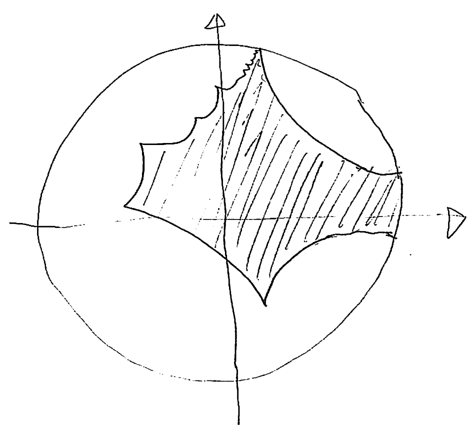
It follows that for any $g \in \mathrm{PSU}(1,1)$, we have

$$\rho(w,0)\begin{Bmatrix}<\\=\\>\end{Bmatrix}\rho(gw,0)\ \text{ according as }\ \begin{cases}w \in \operatorname{ext}(I(g)),\\ w \in I(g),\\ w \in \operatorname{int}(I(g)).\end{cases}$$

In particular, we have

$$⌑(\Gamma;0) = \bigcap_{\gamma \in \Gamma - \mathrm{Stab}_\Gamma(0)} \operatorname{cl}\operatorname{ext} I(\gamma).$$

This description of a Dirichlet domain as the intersection of the exteriors of isometric circles is due to Ford, and so we call a Dirichlet domain in $\mathbf{D}^2$ centered at 0 a *Ford domain*. In section 31.7, we show how this description can be turned into an algorithm for computing the Dirichlet domain $⌑$ for a nice class of Fuchsian groups.



Remark 31.2.10. In the identification $\mathbf{H}^2 \to \mathbf{D}^2$, the preimage of isometric circles corresponds to the corresponding perpendicular bisector; this is the simplification provided by working in $\mathbf{D}^2$ (the map ϕ is a hyperbolic isometry, whereas isometric circles are defined by a Euclidean condition).

31.3 Side pairings and Poincaré's polygon theorem

Continuing with our third and final section focused on Fuchsian groups, we consider applications to the structure of a Fuchsian group Γ, as well as a partial converse.

Let $⌑ = ⌑(\Gamma; z_0)$ be a Dirichlet domain centered at $z_0 \in \mathbf{H}^2$. A consequence of the local finiteness of a Dirichlet domain is the following theorem (Theorem 31.4.2).

Theorem 31.3.1. Γ *is generated by the set*

$$\{\gamma \in \Gamma : ⌑ \cap \gamma ⌑ \neq \emptyset\}.$$

So by Theorem 31.3.1, to find generators, we must look for "overlaps" in the tesselation provided by $⌑$. If $z \in ⌑ \cap \gamma ⌑$ with $\gamma \in \Gamma - \{1\}$, then $z, \gamma z \in ⌑$, so

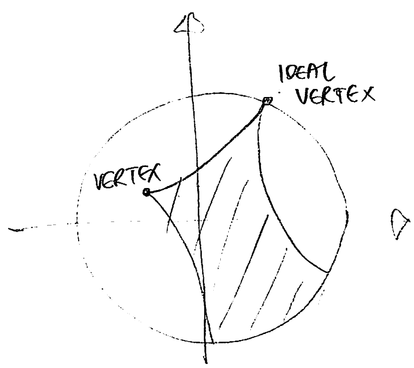
$$\rho(z, z_0) \leq \rho(\gamma z, z_0) \leq \rho(z, z_0) \tag{31.3.2}$$

so equality holds and (viz. Paragraph 31.1.2) we have $z \in \operatorname{bd} ⌑$. Since the boundary of $⌑$ is geodesic, to understand generators we should organize the matching provided along the geodesic boundary of $⌑$.

For convenience, from now on, we work in the unit disc $\mathbf{D}^2$.

31.3.3. A maximal geodesic subset of $⌑$ (of nonzero length) is called a *side*. Equivalently, a side is a nonempty set of the form $⌑ \cap \gamma ⌑$ with $\gamma \in \Gamma - \{1\}$ by (31.3.2), and such a representation is unique.

If two sides intersect in $\mathbf{D}^2$, the point of intersection is called a *vertex* of $⌑$; equivalently, a vertex is a single point of the form $⌑ \cap \gamma ⌑ \cap \gamma' ⌑$ with $\gamma, \gamma' \in \Gamma$. An *ideal vertex* is a point of the closure of $⌑$ in $\mathbf{D}^{2*}$ that is the intersection of the closure of two sides in $\mathbf{D}^{2*}$.



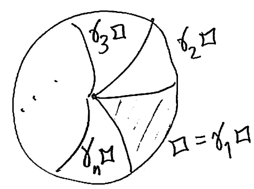
Because $⌑$ is locally finite, there are only finitely many vertices in any compact neighborhood (Exercise 31.7).

31.3.4. We make the following important convention on sides and vertices to simplify the arguments below (at the cost of making their description slightly more complicated): if

$$L = ⌑ \cap \gamma ⌑$$

is a side of $⌑$ and $\gamma^2 = 1$, or equivalently if $\gamma L = L$, then γ fixes the midpoint of L, and we consider L to be the union of *two* sides that meet at the vertex equal to the midpoint.

31.3.5. We now provide a *standard picture* of ⌑ in a neighborhood of a point $w \in \operatorname{bd}$ ⌑. Because ⌑ is locally finite, there is an an open neighborhood of w and finitely many $\gamma_1, \ldots, \gamma_n \in \Gamma$ with $\gamma_1 = 1$ such that $U \subseteq \bigcup_i \gamma_i$⌑ and $w \in \gamma_i$⌑ for all i. Shrinking U if necessary, we may assume that U contains no vertices of ⌑ except possibly for w and intersects no sides of γ_i⌑ except those that contain w. Therefore, we have the following situation:



(31.3.6)

(As a special case, we may have $n = 2$, and then either w is a fixed point of γ_2 or not.)

31.3.7. Let S denote the set of sides of ⌑. We define a labeled equivalence relation on S by

$$P = \{(\gamma, L, L^*) : L^* = \gamma(L)\} \subset \Gamma \times (S \times S). \tag{31.3.8}$$

We say that P is a *side pairing* for P if P induces a partition of S into pairs, and we denote by $G(P)$ the projection of P to Γ. Since $(\gamma, L, L^*) \in P$ implies $(\gamma^{-1}, L^*, L) \in P$, we have that $G(P)$ is closed under inverses.

31.3.9. We will also need to consider an induced relation on the set of vertices. Let $v = v_1$ be a vertex of ⌑. Then by the standard picture, there exist $1 = \gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$ such that if $v_i = \gamma_i v_1$ then $\Gamma v \cap$ ⌑ $= \{v_1, \ldots, v_n\}$. We call $v_1, \ldots, v_n$ a *vertex cycle*, and to such a cycle we write $g_i = \gamma_i^{-1}\gamma_{i+1}$ (indices taken modulo n) and associate the *cycle relation*

$$g_1 g_2 \cdots g_n = (\gamma_1^{-1}\gamma_2)(\gamma_2^{-1}\gamma_3) \cdots (\gamma_n \gamma_1) = 1. \tag{31.3.10}$$

Let $R(P)$ be the finite set of cycle relations arising from Γ-orbits of vertices in ⌑.

Proposition 31.3.11. *A Dirichlet domain ⌑ has a side pairing P, and the set $G(P)$ generates Γ with $R(P)$ a set of defining relations.*

Proposition 31.3.11 is to be interpreted this way: the free group on $G(P)$ modulo the normal subgroup generated by the relations $R(P)$ is isomorphic to Γ via the natural evaluation map.

Proof. If L is a side, then $L =$ ⌑ $\cap\, \gamma$⌑ for a unique γ, and thus

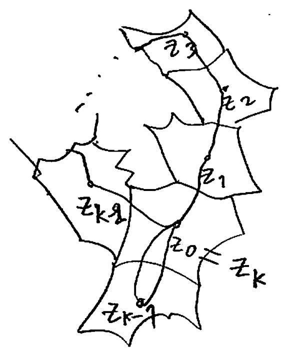
$$\gamma^{-1}L = ⌑ \cap \gamma^{-1}⌑ = L^* \neq L$$

by the convention in Paragraph 31.3.4; by uniqueness, the equivalence class of L contains only L, L^*, so P (31.3.8) is a side pairing.

Let $\Gamma^* \leq \Gamma$ be the subgroup generated by $G(P)$. By Theorem 31.3.1, we need to show that if ⌑ $\cap\, \gamma$⌑ $\neq \emptyset$ then $\gamma \in \Gamma^*$. So let $w \in$ ⌑ $\cap\, \gamma$⌑ with $\gamma \in \Gamma - \{1\}$.

We refer to the standard picture (Paragraph 31.3.5); we have $\gamma = \gamma_j$ for some j. For all $i = 1, \dots, n$, we have ⌑ $\cap\, \gamma_i^{-1}\gamma_{i+1}$⌑ is a side, so $\gamma_i^{-1}\gamma_{i+1} \in G(P)$ is a side pairing element. Since $\gamma_1 = 1$, by induction we find that $\gamma_i \in \Gamma^*$ for all i, so $\gamma = \gamma_j \in \Gamma^*$ as claimed. It follows that, in fact, each cycle relation 31.3.10, we have $g_i = \gamma_i^{-1}\gamma_{i+1} \in G(P)$ so the relation is already a word in $G(P)$.

We now turn to relations. Let $h_1 h_2 \dots h_k = 1$ be a relation with each $h_i \in G(P)$, and let $z_i = h_i z_{i-1}$ for $i = 1, \dots, k$. Exactly because $g_1 \in G(P)$, we have that ⌑ and g_1⌑ share a side, and since ⌑ is connected, we can draw a path $z_0 \to z_1$ through the corresponding side. Continuing in this way, we end up with a path $z_0 \to z_k = z_0$, hence a closed loop.



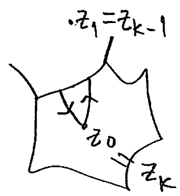
Let V be the intersection of the Γ orbit of the vertices of ⌑ with the interior of the loop; this is a finite set, and we proceed by induction on its cardinality. The proof boils down to the fact that this loop retracts onto the loops around vertices obtained from cycle relations.

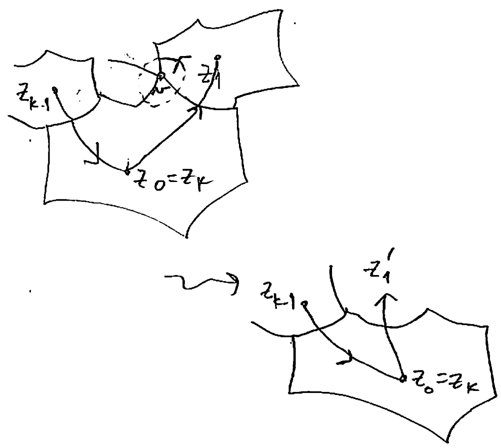
If the path from $z_0 \to z_1$ crosses the same side as the path $z_{k-1} \to z_k = z_0$, then $z_1 = z_{k-1}$ and so

$$g_1 z_0 = z_1 = z_{k-1} = g_{k-1}^{-1} z_k = g_k^{-1} z_0$$

so $g_k^{-1} = g_1$, since $\mathrm{Stab}_\Gamma(z_0) = \{1\}$. Conjugating the relation by $g_k = g_1$ and repeating if necessary, we may assume that $g_k^{-1} \neq g_1$, so $z_{k-1} \neq z_1$; and the set V is conjugated, so it remains the same size. In particular, if V is empty, then this shows that the original relation is conjugate to the trivial relation.

Otherwise, the path $z_0 \to z_1$ crosses a side and there is a unique vertex on this side that is interior to the loop (working counterclockwise). The cycle relation $h_1 \cdots h_k$ around v provides a loop around v starting with $g_1 = h_1 = h_k^{-1} \cdots h_2^{-1}$; substituting this into the original relation, we obtain a new relation with one fewer vertex; the result then follows by induction.





$\square$

In section 31.5, we consider a partial converse to Proposition 31.3.11, due to Poincaré: given a convex hyperbolic polygon with a side pairing that satisfies certain conditions, there exists a Fuchsian group Γ with the given polygon as a fundamental domain.

31.4 Dirichlet domains

In this section we consider the construction of Dirichlet fundamental domains in a general context. Let (X, ρ) be a complete, locally compact geodesic space. In particular, X is connected, and by the theorem of Hopf–Rinow, closed balls (of finite radius) in X are compact.

Let Γ be a discrete group of isometries acting properly on X. Right from the get go, we prove our first important result: we exhibit generators for a group based on a fundamental set with a basic finiteness property.

Definition 31.4.1. Let $A \subseteq X$. We say A is *locally finite* for Γ if for each compact set $K \subset X$, we have $\gamma K \cap A \neq \emptyset$ for only finitely many $\gamma \in \Gamma$.

The value of a locally finite fundamental set is explained by the following theorem.

Theorem 31.4.2. *Let* ¤ *be a locally finite fundamental set for* Γ. *Then* Γ *is generated by the set*

$$\{\gamma \in \Gamma : \text{¤} \cap \gamma\text{¤} \neq \emptyset\}. \tag{31.4.3}$$

Proof. Let $\Gamma^* \leq \Gamma$ be the subgroup of Γ generated by the elements (31.4.3). We want to show $\Gamma^* = \Gamma$.

For any $x \in X$, by Theorem 31.4.16, there exists $\gamma \in \Gamma$ such that $\gamma x \in \text{¤}$. If there is another $\gamma' \in \Gamma$ with $\gamma' x \in \text{¤}$, then

$$\gamma' x \in \text{¤} \cap \gamma'\gamma^{-1}\text{¤}$$

so $\gamma'\gamma^{-1} \in \Gamma^*$ and in particular $\Gamma^*\gamma = \Gamma^*\gamma'$. In this way, we define a map

$$\begin{aligned} f : X &\to \Gamma^* \backslash \Gamma \\ x &\mapsto \Gamma^*\gamma \end{aligned}$$

for any $\gamma \in \Gamma$ such that $\gamma x \in ⌑$.

We now show that f is locally constant. Let $x \in X$. Since $⌑$ is locally finite, for any compact neighborhood $K \ni x$ we can write $K \subseteq \bigcup_i \gamma_i ⌑$ with a finite union, and by shrinking K we may assume that $x \in \gamma_i ⌑$ for all i. In particular, $f(x) = \Gamma^*\gamma_i^{-1}$ for any i. But then if $y \in K$, then $y \in \gamma_i ⌑$ for some i, so $f(y) = \Gamma^*\gamma_i^{-1} = f(x)$. Thus f is locally constant.

But X is connected so any locally constant function is in fact constant, so f takes only the value Γ^*. So now let $\gamma \in \Gamma$ and let $x \in ⌑$. Then

$$\Gamma^* = f(x) = f(\gamma^{-1}x) = \Gamma^*\gamma$$

so $\gamma \in \Gamma^*$, and the proof is complete. $\square$

We now seek a locally finite fundamental set with other nice properties: we will choose in each Γ-orbit the closest points to a fixed point $x_0 \in X$. So we first must understand the basic local properties of intersections of these half-spaces (as in Paragraph 31.1.2).

31.4.4. For $x_1, x_2 \in X$, define the closed *Leibniz half-space*

$$H(x_1, x_2) = \{x \in X : \rho(x, x_1) \le \rho(x, x_2)\}. \qquad (31.4.5)$$

If $x_1 = x_2$, then $H(x_1, x_2) = X$. If $x_1 \neq x_2$, then $H(x_1, x_2)$ consists of the set of points as close to x_1 as x_2, so

$$\operatorname{int} H(x_1, x_2) = \{x \in X : \rho(x, x_1) < \rho(x, x_2)\}. \qquad (31.4.6)$$

and

$$\operatorname{bd} H(x_1, x_2) = L(x_1, x_2) = \{x \in X : \rho(x, x_1) = \rho(x, x_2)\}$$

is called the *hyperplane bisector* (or *equidistant hyperplane* or *separator*) between x_1 and x_2.

Remark 31.4.7. In this generality, unfortunately hyperplane bisectors are not necessarily geodesic (Exercise 31.9).

Definition 31.4.8. A set $A \subseteq X$ is *star-shaped* with respect to $x_0 \in A$ if for all $x \in A$, the geodesic between x and x_0 belongs to A.

A set $A \subseteq X$ that is star-shaped is path connected, so connected.

Lemma 31.4.9. *A Leibniz half-plane $H(x_1, x_2)$ is star-shaped with respect to x_1.*

Proof. Let $x \in H(x_1, x_2)$ and let y be a point along the unique geodesic from x to x_1. Then

$$\rho(x_1, y) + \rho(y, x) = \rho(x_1, x).$$

If $y \notin H(x_1, x_2)$, then $\rho(x_2, y) < \rho(x_1, y)$, and so

$$\rho(x_2, x) \leq \rho(x_2, y) + \rho(y, x) < \rho(x_1, y) + \rho(y, x) = \rho(x_1, x)$$

contradicting that $x \in H(x_1, x_2)$. So $y \in H(x_1, x_2)$ as desired. ☐

Now let $x_0 \in X$.

Definition 31.4.10. The *Dirichlet domain* for Γ centered at $x_0 \in X$ is

$$\text{⌑}(\Gamma; x_0) = \{z \in \mathbf{H}^2 : \rho(x, x_0) \leq \rho(\gamma x, x_0) \text{ for all } \gamma \in \Gamma\}.$$

We often abbreviate $\text{⌑}(x_0) = \text{⌑}(\Gamma; x_0)$.

31.4.11. Since $\rho(\gamma x, x_0) = \rho(x, \gamma^{-1} x_0)$, we have

$$\text{⌑}(x_0) = \bigcap_{\gamma \in \Gamma} H(x_0, \gamma^{-1} x_0); \tag{31.4.12}$$

each half-space is closed and star-shaped with respect to x_0, so the same is true of $\text{⌑}(x_0)$. In particular, $\text{⌑}(x_0)$ is connected.

A Dirichlet domain satisfies a basic finiteness property, as follows.

Lemma 31.4.13. *If $A \subset X$ is any bounded set, then $A \subseteq H(\gamma; x_0)$ for all but finitely many $\gamma \in \Gamma$.*

In particular, for any $x \in X$ we have $x \in H(\gamma; x_0)$ for all but finitely many $\gamma \in \Gamma$.

Proof. Since A is bounded, we have

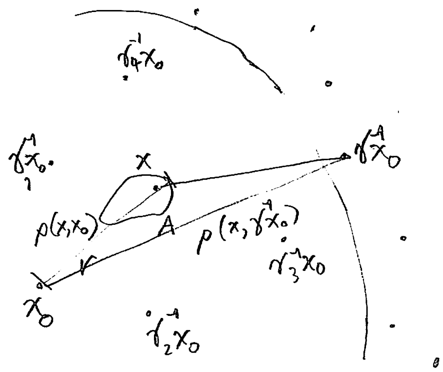
$$\sup(\{\rho(x, x_0) : x \in A\}) = r < \infty.$$

By Theorem 28.3.12, the orbit Γx_0 is discrete and $\#\operatorname{Stab}_\Gamma(x_0) < \infty$; since closed balls are compact by assumption, there are only finitely many $\gamma \in \Gamma$ such that

$$\rho(\gamma x_0, x_0) = \rho(x_0, \gamma^{-1} x_0) \leq 2r$$

and for all remaining $\gamma \in \Gamma$ and all $x \in A$, we have

$$\rho(x, \gamma^{-1} x_0) \geq \rho(x_0, \gamma^{-1} x_0) - \rho(x, x_0) > 2r - r = r \geq \rho(x, x_0)$$



so $x \in H(\gamma; x_0)$. $\square$

31.4.14. Arguing in a similar way as in Lemma 31.4.13, one can show (Exercise 31.8): if K is any compact set, then $K \cap L(\gamma; x_0) \neq \emptyset$ for only finitely many $\gamma \in \Gamma$.

Lemma 31.4.15. *We have*

$$\operatorname{int} \mathcal{D}(x_0) = \{x \in \mathcal{D} : \rho(x, x_0) < \rho(\gamma x, x_0) \text{ for all } \gamma \in \Gamma - \operatorname{Stab}_\Gamma(x_0)\}$$

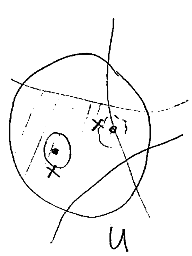
and

$$\operatorname{bd} \mathcal{D}(x_0) = \{x \in \mathcal{D} : \rho(x, x_0) = \rho(\gamma x, x_0) \text{ for some } \gamma \in \Gamma - \operatorname{Stab}_\Gamma(x_0)\}.$$

Proof. Let $x \in \mathcal{D}$, and let $U \ni x$ be a bounded open neighborhood of x. By Lemma 31.4.13, we have $U \subseteq H(\gamma; x_0)$ for all but finitely many $\gamma \in \Gamma$, so

$$U \cap \mathcal{D} = U \cap \bigcap_i H(x_0, \gamma_i^{-1} x_0)$$

the intersection over finitely many $\gamma_i \in \Gamma$ with $\gamma_i \notin \operatorname{Stab}_\Gamma(x_0)$.



Thus

$$U \cap \operatorname{int}(\mathcal{D}) = U \cap \bigcap_i \operatorname{int} H(x_0, \gamma_i^{-1} x_0).$$

The lemma then follows from (31.4.6). $\square$

The first main result of this chapter is the following theorem.

Theorem 31.4.16. *Let $x_0 \in X$, and suppose $\mathrm{Stab}_\Gamma(x_0) = \{1\}$. Then $\Box(\Gamma; x_0)$ is a locally finite fundamental set for Γ that is star-shaped with respect to x_0 and whose boundary consists of hyperplane bisectors.*

Specifically, in any bounded set A, by Lemma 31.4.13 we have

$$A \cap \mathrm{bd}\,\Box(\Gamma; x_0) \subseteq \bigcup_i L(x_0, \gamma_i x_0)$$

for finitely many $\gamma_i \in \Gamma - \{1\}$.

Proof. Abbreviate $\Box = \Box(x_0)$. We saw that $\Box$ is (closed and) star-shaped with respect to x_0 in Paragraph 31.4.11.

Now we show that $X = \bigcup_\gamma \gamma\Box$. Let $x \in X$. The orbit Γx is discrete, so the distance

$$\rho(\Gamma x, x_0) = \inf(\{\rho(\gamma x, x_0) : \gamma \in \Gamma\}) \tag{31.4.17}$$

is minimized at some point $\gamma x \in \Box$ with $\gamma \in \Gamma$. Thus $\Box(x_0)$ contains at least one point from every Γ-orbit, and consequently .

We now refer to Lemma 31.4.15. Since X is complete, this lemma implies that $\mathrm{cl}(\mathrm{int}(\Box)) = \Box$. And $\mathrm{int}(\Box) \cap \mathrm{int}(\gamma\Box) = \emptyset$ for all $\gamma \in \Gamma - \{1\}$, because if $x, \gamma x \in \mathrm{int}(\Box)$ with $\gamma \neq 1$ then

$$\rho(x, x_0) < \rho(\gamma x, x_0) < \rho(\gamma^{-1}(\gamma x), x_0) = \rho(x, x_0), \tag{31.4.18}$$

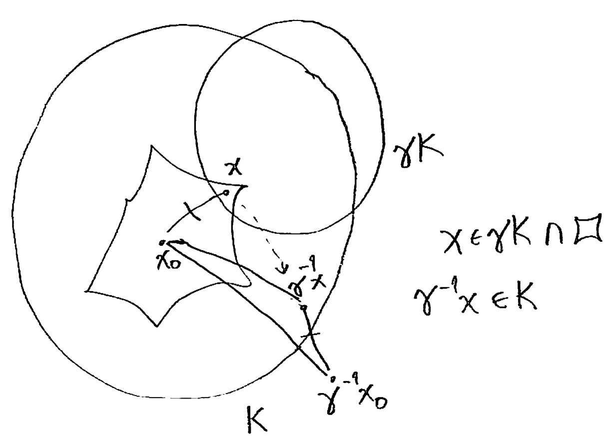
a contradiction.

Finally, we show that X is locally finite. It suffices to check this for a closed disc $K \subseteq X$ with center x_0 and radius $r \in \mathbb{R}_{\geq 0}$. Suppose that γK meets $\Box$ with $\gamma \in \Gamma$; then by definition there is $x \in \Box$ such that $\rho(x_0, \gamma^{-1}x) \leq r$. Then

$$\rho(x_0, \gamma^{-1}x_0) \leq \rho(x_0, \gamma^{-1}x) + \rho(\gamma^{-1}x, \gamma^{-1}x_0) \leq r + \rho(x, x_0).$$

Since $x \in \Box$, we have $\rho(x, x_0) \leq \rho(\gamma^{-1}x, x_0) \leq r$, so

$$\rho(x_0, \gamma^{-1}x_0) \leq r + r = 2r.$$



For the same reason as in Lemma 31.4.13, this can only happen for finitely many $\gamma \in \Gamma$. $\square$

31.5 Hyperbolic Dirichlet domains

We now specialize to the case $X = \mathcal{H}$ where $\mathcal{H} = \mathbf{H}^2$ or $\mathcal{H} = \mathbf{H}^3$ with volume μ; then Γ is a Fuchsian or Kleinian group, respectively.

Definition 31.5.1. A *fundamental domain* for $\Gamma \circlearrowright X$ is a connected fundamental set $⌑ \subseteq X$ such that $\mu(\operatorname{bd} ⌑) = 0$.

We now turn to Dirichlet domains in this context.

31.5.2. On the one hand, the hypothesis that $\operatorname{Stab}_\Gamma(z_0) = \{1\}$ is a very mild hypothesis, as follows. If $\mathcal{H} = \mathbf{H}^2$, then in any compact set K, there are only finitely many points $z \in K$ such that $\operatorname{Stab}_\Gamma(z) \neq \{1\}$: indeed, there are only finitely many $\gamma \in \Gamma$ such that $\gamma K \cap K \neq \emptyset$, and any such $\gamma \neq 1$ has at most one fixed point in $\mathbf{H}^2$ (Lemma 27.3.5). Similarly, if $\mathcal{H} = \mathbf{H}^3$, in any compact set K, the set of points z with $\operatorname{Stab}_\Gamma(z) \neq \{1\}$ is a finite set of points and geodesic axes.

That being said, we can then prove a slightly stronger and more useful version of Theorem 31.4.16, as follows. If $\Gamma_0 = \operatorname{Stab}_\Gamma(z_0)$ is nontrivial, we modify the Dirichlet domain by intersecting with a fundamental set for Γ_0; the simplest way to do this is just to choose another point which is not fixed by any element of Γ_0 and intersect.

Theorem 31.5.3. *Let $z_0 \in \mathcal{H}$, let $\Gamma_0 = \operatorname{Stab}_\Gamma(z_0)$, and let $u_0 \in \mathcal{H}$ be such that $\operatorname{Stab}_{\Gamma_0}(u_0) = \{1\}$. Then*

$$⌑(\Gamma; z_0) \cap ⌑(\Gamma_0; u_0)$$

is a connected, convex, locally finite fundamental domain for Γ with geodesic boundary in $\mathcal{H}$.

Proof. Abbreviate

$$⌑ = ⌑(\Gamma; z_0) \cap ⌑(\Gamma_0; u_0).$$

First, we show that $z_0 \in ⌑$: indeed, $z_0 \in ⌑(\Gamma; z_0)$, and by Theorem 31.4.16, $⌑(\Gamma_0; u_0)$ is a fundamental set for Γ_0 so there exists $\gamma_0 \in \Gamma_0$ such that $\gamma_0 u_0 = u_0 \in ⌑(\Gamma_0; u_0)$.

Now we show that $⌑$ is a fundamental set for Γ. First we show $\mathcal{H} = \bigcup_\gamma \gamma ⌑$. Let $z \in \mathcal{H}$, and let $\gamma \in \Gamma$ be such that $\rho(\gamma z, z_0)$ is minimal as in (31.4.17). Let $\gamma_0 \in \Gamma_0$ such that $\gamma_0(\gamma z) \in ⌑(\Gamma_0; u_0)$. Then

$$\rho(\gamma_0(\gamma z), z_0) = \rho(\gamma z, z_0)$$

so $\gamma_0 \gamma z \in ⌑$. And $\operatorname{int}(⌑) \cap \operatorname{int}(\gamma ⌑) = \emptyset$ for all $\gamma \in \Gamma - \{1\}$, because if $z, \gamma z \in \operatorname{int}(⌑)$ with $\gamma \neq 1$, then either $\gamma \notin \Gamma_0$ in which case we obtain a contradiction as in (31.4.18), or $\gamma \in \Gamma_0 - \{1\}$ and then we have a contradiction from the fact that $⌑(\Gamma_0; u_0)$ is a fundamental set.

We conclude by proving the remaining topological properties of $⌑$. We know that $⌑$ is locally finite, since it is the intersection of two locally finite sets. We saw in Paragraph 31.1.2 that the Leibniz half-spaces in $\mathbf{H}^2$ are convex with geodesic boundary, and the same is true in $\mathbf{H}^3$ by Exercise 30.8. It follows that $⌑$ is convex, as the intersection of convex sets. Thus

$$\operatorname{bd} ⌑ \subseteq \bigcup_{\gamma \in \Gamma - \{1\}} L(z_0, \gamma^{-1} z_0)$$

is geodesic and measure zero, since $L(z_0, \gamma^{-1} z_0)$ intersects a compact set for only finitely many γ by Paragraph 31.4.14. □

We now turn to a partial converse for Theorem 31.5.3 for $\mathbf{H}^2$; we need one additional condition. Let $⌑$ be a convex (finite-sided) hyperbolic polygon equipped with a side pairing P.

31.5.4. For a vertex v of $⌑$, let $\vartheta(⌑, v)$ be the interior angle of $⌑$ at v. We say that $⌑$ *satisfies the cycle condition* if for vertex v of $⌑$, there exists $e \in \mathbb{Z}_{>0}$ such that

$$\sum_{\substack{\gamma \in \Gamma \\ \gamma v \in ⌑}} \vartheta(⌑, \gamma v) = \frac{2\pi}{e}.$$

Put another way, $⌑$ satisfies the cycle condition if the sum of the interior angles for a Γ-orbit of vertices as in the standard picture is an integer submultiple of 2π.

Lemma 31.5.5. *Let $⌑$ be a Dirichlet domain. Then $⌑$ satisfies the cycle condition.*

Proof. Let v be a vertex of $⌑$ and let $\Gamma v = \{v_1, \dots, v_n\}$ with $\delta_i v_i = v_1 = v$. Then $\gamma_i ⌑$ has v as a vertex, and

$$\vartheta(\delta_i ⌑, v) = \vartheta(⌑, v_i) = \vartheta(⌑, \delta_i v).$$

Referring to (31.3.6), we see that

$$\{\gamma_i\}_i = \bigcup_i \operatorname{Stab}_\Gamma(v) \delta_i.$$

Letting $e = \# \operatorname{Stab}_\Gamma(v)$, since $\operatorname{Stab}_\Gamma(v)$ acts (locally) by rotation around v, we conclude that

$$2\pi = e \sum_{i=1}^{n} \vartheta(⌑, v_i)$$

and so the cycle condition is satisfied. □

Theorem 31.5.6 (Poincaré)**.** *Let $⌑$ be a convex (finite-sided) hyperbolic polygon equipped with a side-pairing P. Suppose that $⌑$ satisfies the cycle condition.*

Then the group $\Gamma = \langle G(P) \rangle \leq \mathrm{PSL}_2(\mathbb{R})$ generated by side pairing elements is a Fuchsian group, and $⌑$ is a fundamental domain for Γ.

Unfortunately, it is beyond the scope of this book to give a complete proof of Theorem 31.5.6, but see Paragraph 31.8.2 for further references, including generalizations.

31.6 Coset tables

[[Discuss the Reidemeister–Schreier algorithm?]]

31.7 Algorithmic aspects

We now show how one can explicitly compute the Dirichlet or Ford domain ⌑ for finitely generated Fuchsian and Kleinian groups.

31.8 Extensions and further reading

A Fuchsian group with cofinite area if finitely generated by a result of Siegel [Kat92, Theorem 4.1.1], [GG90, §1].

Dirichlet domains are sometimes also called Voronoi domains, because of the link to Voronoi theory.

31.8.1. A discussion of side pairings can be found in Beardon [Bea95, Theorem 9.3.3] and Katok [Kat92, Theorem 3.5.4].

31.8.2. Poincaré's theorem (Proposition 31.3.11) states that a finite-sided hyperbolic polygon $⌑ \subseteq \mathbf{D}^2$ equipped with a side pairing P that satisfies the cycle condition is a fundamental domain for the group generated by $G(P)$. This is proven by Beardon [Bea95, Theorem 9.8.4] and the accompanying exercises: the condition that $\mu(⌑) < \infty$ ensures that any vertex which lies on the circle at infinity is fixed by a hyperbolic element [GG90, §1]. One must verify Beardon's condition (A6) [Bea95, p. 246] or (A6)′ [Bea95, p. 249], which formalizes the equivalent angle condition (g) given by Maskit [Mas71, p. 223]. This statement extends to a larger class of polygons (see Beardon [Bea95, §9.8]).

A *domain* in topology is sometimes taken to be an open connected set; one also sees *closed domains*, and our fundamental domains are taken to be of this kind.

31.8.3. The characterization of the domain $⌑(\Gamma; 0) \subseteq \mathbf{D}^2$ in Paragraph 31.2.9 is originally attributed to Ford [For1972, Theorem 7, §20].

Exercises

31.1. Describe the Dirichlet domain $⌑(z)$ for an arbitrary $z \in \mathbf{H}^2$ with $\operatorname{Im} z > 1$.

31.2. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ (cf. section 30.5). Show that

$$⌑(\Gamma; 2j) = \{z = x + yj \in \mathbf{H}^3 : |\operatorname{Re} x|, |\operatorname{Im} x| \leq 1/2 \text{ and } \|z\| \geq 1\}$$

and that

$$\mathrm{Stab}_\Gamma(2j) = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

so $⌑(\Gamma; 2j)$ is a union of two copies of a fundamental set for Γ.

31.3. Let Γ be the cyclic Fuchsian group generated by the isometry $z \mapsto 4z$, represented by $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$. Give an explicit description of the Dirichlet domain $⌑(\Gamma; i) \subset \mathbf{H}^2$ and its image $⌑(\Gamma; 0) \subset \mathbf{D}^2$ with $i \mapsto 0$.

31.4. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ with $gi \neq i$, show that $\|g\|^2 > 2$ and that the perpendicular bisector between i and gi is the half-circle of radius $\dfrac{\|g\|^2 - 2}{(a^2 + c^2 - 1)^2}$ centered at $\dfrac{ab + cd}{a^2 + c^2 - 1} \in \mathbb{R}$, where $\|g\|^2 = a^2 + b^2 + c^2 + d^2$.

31.5. Let

$$g = \begin{pmatrix} \overline{d} & \overline{c} \\ c & d \end{pmatrix} \in \mathrm{PSU}(1, 1) \circlearrowright \mathbf{D}^2$$

with $c, d \in \mathbb{C}$ satisfying $|d|^2 - |c|^2 = 1$. Show directly that $|gw| = |w|$ for $w \in \mathbf{D}^2$ if and only if

$$|cw + d| = 1$$

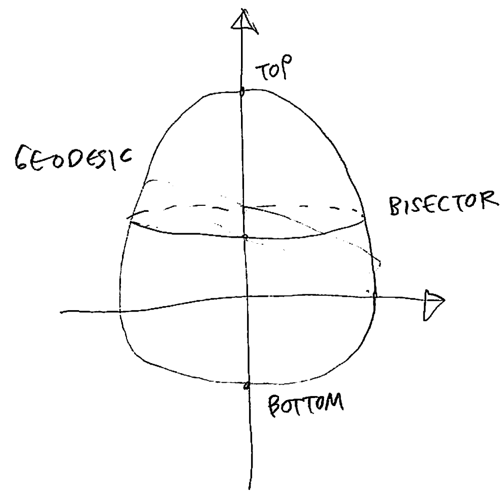
by expanding out and simplifying.

31.6. Show that for any $g \in \mathrm{PSU}(1, 1)$, we have $gI(g) = I(g^{-1})$, where $I(g)$ is the isometric circle of g. (Equivalently, show that if $g \in \mathrm{PSL}_2(\mathbb{R})$ that $gL(g; z_0) = L(g^{-1}; z_0)$ for any $z_0 \in \mathbf{H}^2$.)

31.7. Let $⌑$ be a Dirichlet domain for a Fuchsian group Γ. Show that in any compact set, there are only finitely many vertices of $⌑$.

31.8. Let Γ be a discrete group of isometries acting properly on a locally compact, complete metric space X, and let $x_0 \in X$. Recall the definition of $H(\gamma; x_0)$ (31.4.5) for $\gamma \in \Gamma$ and $L(\gamma; z_0) = \mathrm{bd}\, H(\gamma; z_0)$. Show that if K is any compact set, then $K \cap L(\gamma; x_0) \neq \emptyset$ for only finitely many $\gamma \in \Gamma$.

31.9. Consider the egg of revolution, a surface of revolution obtained from convex curves with positive curvature as in the following picture:

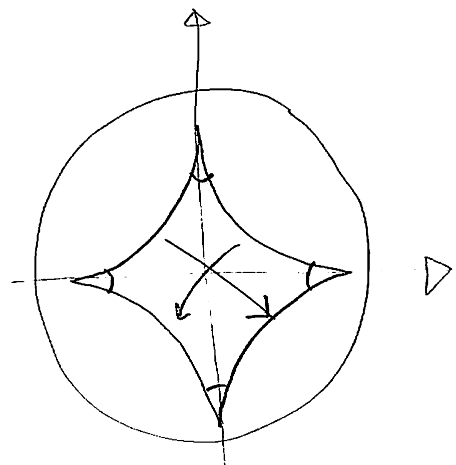


An egg of revolution has the structure of a geodesic space with the induced metric from $\mathbb{R}^3$. Show that the separator between the top and bottom, a circle of revolution, is *not* geodesic. [In fact, Clairaut's relation shows that the geodesic joining two points in the same circle of revolution above crest in the x-axis never lies in this circle of revolution.]

31.10. In this exercise, we consider a Fuchsian group constructed from a regular quadrilateral.

a) Show that for every $e \geq 2$, there exists a regular quadrilateral ⌑ $\subset \mathbf{D}^2$, unique up to isometry, with interior angle $\pi/(2e)$.

Conclude from Poincaré's theorem that there is a Fuchsian group, unique up to conjugation in $\mathrm{PSL}_2(\mathbb{R})$, with fundamental domain ⌑ and side pairing as follows.



b) Find explicit generators and relations for this group when $e = 2$.